East Asian Math J 17(2001), No. 2, pp 275-286

SOME REMARKS ON SKEW POLYNOMIAL RINGS OVER REDUCED RINGS

HONG KEE KIM

ABSTRACT In this paper, a skew polynomial ring $R[x,\alpha]$ of a ring R with a monomorphism α are investigated as follows. For a reduced ring R, assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R Then (i) $R[x,\alpha]$ is a reduced ring, (ii) a ring R is Baer (resp. quasi-Baer, p.q -Baer, a p p -ring) if and only if the skew polynomial ring $R[x,\alpha]$ is Baer (resp. quasi-Baer, p.q -Baer, a p.p -ring)

0. Introduction

Let R be an associative ring with unity throughout this paper. The following notations will be preserved: Let α be an endomorphism of a ring R. An α − *derivation* of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \alpha(a)\delta(b) + \delta(a)b$ for all $a, b \in R$. The Ore extension $R[x, \alpha, \delta]$ is the ring of polynomials in x over R with the usual addition and with new multiplication by $xa = \alpha(a)x + \delta(a)$ for each $a \in R$. If $\delta = 0$, we write $R[x;\alpha]$ for $R[x;\alpha,\delta]$ and is called an *Ore extension* of *endomorphism type* (also called a *skew polynomial* ring). While if $\alpha = 1$, we write $R[x;\delta]$ for $R[x;1,\delta]$ and is called an *Ore extension* of *derivation type* (also called a *differential polynomial* ring). Moreover, $R[[x;\alpha]]$ is called a skew power series ring.

Received March 3, 2001. Revised September 29, 2001.

2000 Mathematics Subject Classification 16D10, 16D15, 16S36

Key words and phrases reduced ring, skew polynomial ring, Baer ring, quasi-Baer ring, p.q -Baer ring, p p -ring.

This work was supported by Korea Research Foundation Grant (KRF- 2000-015-DP0002)

Also, we recalled that R is a *reduced* ring if it has no nonzero nilpotent elements.

In this paper, if R is a reduced ring and satisfies a condition, then we will prove the following:

(1) the skew polynomial ring $R[x;\alpha]$ of a ring R is a reduced ring.

(2) a ring R is Baer (resp. quasi-Baer, p.q.-Baer, p.p.-ring) if and only if the skew polynomial ring $R[x;\alpha]$ is Baer (resp. quasi-Baer, p.q.-Baer, p.p.-ring).

1. Properties of reduced rings

First, we have the following well- known fact:

THEOREM 1.1. *Let R be an integral domain with a monomorphism α. Then the skew polynomial ring $R[x;\alpha]$ is an integral domain.*

PROOF See [5, p16].

In this case, the skew polynomial ring $R[x;\alpha]$ is a reduced ring. Also, we have the following well-known fact:

THEOREM 1.2 *Let α be an inner automorphism of a ring R induced by an invertible element c (i.e. $\alpha(r) = c^{-1}rc$ for all $r \in R$) and $R[x;\alpha]$ the Ore extension of automorphism type. Then the polynomial ring $R[x]$ is isomorphic to $R[x;\alpha]$.*

In this case, if R is a reduced ring, then the skew polynomial ring $R[x;\alpha]$ is a reduced ring.

There exists an example that the skew polynomial ring $R[x;\alpha]$ is not a reduced ring even though R is a reduced ring with an automorphism α of R.

EXAMPLE 1 3 *Let F be a field and $R = F \times F$ with an automorphism α given by $\alpha(a,b) = (b,a)$ for all $(a,b) \in R$. Then R is a reduced ring. In this case, the skew polynomial ring $R[x;\alpha]$ is not a reduced ring because $(1,0)x$ $(\neq 0) \in R[x;\alpha]$ but $(1,0)x(1,0)x = 0$.*

So, under what conditions of a reduced ring R and α an endomorphism of a ring R, is the skew polynomial ring $R[x;\alpha]$ a reduced ring?

We have the following Lemma:

LEMMA 1 4 *Let R be a reduced ring Then for all a, b, c, and $d \in R$,*

(1) *$ab = 0$ if and only if $ba = 0$;*

(2) *If $ab = 0$ and $cb + ad = 0$, then $cb = ad = 0$;*

PROOF (1) is clear.

(2) If $ab = 0$ and $cb + ad = 0$, then $0 = (cb + ad)a = c(ba) + (ad)a = ada$ and so $ad = 0$. Hence $cb = 0$.

We recalled the well-known fact without proof:

PROPOSITION 1 5 *If S is a multiplicative subset(i.e. $a, b \in S$ implies $ab \in S$) of a ring R which is disjoint from an ideal K of R, then there exists an ideal P which is maximal in the set of all ideals of R disjoint from S and containing K. Furthermore, any such an ideal P is a prime ideal.*

Using Proposition 1.5, we will prove the following:

LEMMA 1 6 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then, for all $a, b \in R$, $ab = 0$ if and only if $a\alpha^k(b) = 0$ for $k = 1, 2, \ldots$.*

PROOF. ($\Longrightarrow$) Suppose that there exists a positive integer k such that $a\alpha^k(b) \neq 0$. Then $(a\alpha^k(b))^n \neq 0$ for $n = 1, 2, \ldots$ because R is a reduced ring. Put $S = \{(a\alpha^k(b))^n | n = 0, 1, 2, \ldots\}$ and consider the set $\Gamma = \{I \lhd R | S \cap I = \phi\}$ Then S is a multiplicative set with $S \cap \{0\} = \phi$ and define a partial order relation $\preceq$ on the set Γ by $I_1 \preceq I_2 \Leftrightarrow I_1 \subset I_2$ for any $I_1, I_2 \in \Gamma$. So there exists a prime ideal J of R such that $S \cap J = \phi$ by the Proposition 1.5. Of course, J contains a minimal prime ideal P in R. Since $ab = 0$ and R is reduced, $aRb = \{0\} \subseteq P$ and so either $a \in P$ or $b \in P$

If $a \in P$, then $a\alpha^k(b) \in P$ and so it is a contradiction to $S \cap J = \phi$.

If $b \in P$, then $\alpha^k(b) \in \alpha(P) \subseteq P$ and hence $a\alpha^k(b) \in P$. Thus it is a contradiction to $S \cap J = \phi$. Therefore, $a\alpha^k(b) = 0$ for $k = 1, 2, \ldots$.

($\Longleftarrow$) If $ab \neq 0$, then $\alpha^k(ab) \neq 0$ because α is a monomorphism and hence $(\alpha^k(a)\alpha^k(b))^n \neq 0$ for $n = 1, 2, \ldots$ because R is a reduced ring. Put $S = \{(\alpha^k(a)\alpha^k(b))^n | n = 0, 1, 2, \ldots\}$ and consider the set $\Gamma = \{I \lhd R | S \cap I = \phi\}$. According to the previous method and Proposition 1.5,

there exists a prime ideal J of R such that $S \cap J = \phi$. Of course, J contains a minimal prime ideal P in R. Since $a\alpha^k(b) = 0$ and R is reduced, $aR\alpha^k(b) = \{0\} \subseteq P$ and so either $a \in P$ or $\alpha^k(b) \in P$.
If $a \in P$, then $\alpha^k(a) \in \alpha(P) \subset P$ and $\alpha^k(a)\alpha^k(b) \in P$. So it is a contradiction to $S \cap J = \phi$.
If $\alpha^k(b) \in P$, then $\alpha^k(a)\alpha^k(b) \in P$ and also it is a contradiction to $S \cap J = \phi$. Thus $ab = 0$.

We will use the similar method of proof in [1].

PROPOSITION 1 7. *Let R be a reduced ring with a monomorpism α of R and let f and $g \in R[x;\alpha]$ with $f = \sum_{i=0}^{n} a_i x^i$, $g = \sum_{i=0}^{m} b_i x^i$. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $fg = 0$ if and only if $a_i b_j = 0$ for all i and j $(0 \le i \le n, 0 \le j \le m)$.*

PROOF. Suppose that $fg = 0$ and $m = n$ without loss of generality. Then we get the equations;

$$a_0 b_0 = 0 \cdots (A_0)$$
$$a_0 b_1 + a_1\alpha(b_0) = 0 \cdots (A_1)$$
$$\cdots$$
$$a_0 b_n + a_1\alpha(b_{n-1}) + \cdots + a_n\alpha^n(b_0) = 0 \cdots (A_n).$$

By Lemma 1.4-(1), $a_0b_0 = 0 \Leftrightarrow b_0a_0 = 0$. From $b_0 \times (A_1)$, $b_0a_0b_1 + b_0a_1\alpha(b_0) = 0$ implies $b_0a_1\alpha(b_0) = 0$. By Lemma 1.6, $b_0a_1b_0 = 0$ and so $b_0a_1 = a_1b_0 = 0$. By continuing in this way, we have $a_ib_0 = 0$ for $i = 0, 1, ..., n$ and also $a_i\alpha^k(b_0) = 0$ for $i = 0, 1, ..., n$ and $k = 0, 1, ..., n$ by Lemma 1.6. Thus the original equations $(A_0), (A_1), ..., (A_n)$ reduce to the equations;

$$a_0 b_1 = 0 \cdots (B_1)$$
$$a_0 b_2 + a_1\alpha(b_1) = 0 \cdots (B_2)$$
$$\cdots$$
$$a_0 b_n + a_1\alpha(b_{n-1}) + \cdots + a_{n-1}\alpha^{n-1}(b_1) = 0 \cdots (B_n).$$

Again using the fact that $a_0b_1 = 0$ implies $b_1a_0 = 0$, we conclude from the second equations $(B_1), ..., (B_n)$ that $a_1b_1 = 0$ and then similarly that $a_ib_1 = 0$ for $i = 1, ..., n$. Continuing this process, $a_ib_j = 0$ for $i, j = 0, 1, ..., n$.

Conversely, if $a_ib_j = 0$ for all i and j $(0 \leq i, j \leq m)$, then $a_i\alpha^k(b_j) = 0$ for all i, j and k $(0 \leq i, j, k \leq m)$ by Lemma 1.6 and hence $fg = 0$.

By Mathematical Induction, we have

COROLLARY 1 8. *Let R be a reduced ring with a monomorpism α and let f and $g \in R[[x;\alpha]]$ with $f = \sum_{i=0}^{\infty} a_ix^i, g = \sum_{i=0}^{\infty} b_ix^i$. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $fg = 0$ if and only if $a_ib_j = 0$ for all i and j $(i, j = 0, 1, 2, ...)$.*

COROLLARY 1 9 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then R is a reduced ring if and only if $R[x;\alpha]$ is a reduced ring.*

PROOF Suppose that a ring R is reduced and $f^2 = 0$ for any $f = a_0 + a_1x + \cdots + a_nx^n \in R[x;\alpha]$. Then by Proposition 1.7, $a_ia_j = 0$ for all i, j $(0 \leq i, j \leq n)$. In particular, $a_i^2 = 0$ for all i. Since R is reduced, $a_i = 0$ for all i. Hence $f = 0$ and so $R[x;\alpha]$ is a reduced ring. The converse is clear.

By Corollary 1.8, we have

COROLLARY 1 10 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then R is a reduced ring if and only if the skew power series ring $R[[x;\alpha]]$ is a reduced ring.*

REMARK. *In the Corollary 1.9, the assumption that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R is not superfluous by the Example 1.3.*

2. Properties of Baer rings and generalizations

In this section, we will show the following: For a reduced ring R with a monomorpism α, assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then a ring R is Baer (resp. quasi-Baer, p.q.-Baer, p.p.-ring) if and only if the skew polynomial ring $R[x;\alpha]$ is Baer (resp. quasi-Baer, p.q.-Baer, p.p.-ring). So we recalled some definitions.

A ring R is called (*quasi-*) *Baer* if the right annihilator of every ((right) ideal) nonempty subset of R is generated by an idempotent. In [3], a ring R is called a *right* (resp. *left*) *principally quasi-Baer* (or simply *right* (resp. *left*) *p.q.-Baer*) if the right (resp. left) annihilator of a principal right (resp. left) ideal is generated by an idempotent. A ring R is called a *p.q.-Baer ring* if it is both right and left p.q.-Baer. Another generalization of Baer ring is the p.p.-ring. A ring R is called a *right* (resp. *left*) *p.p.- ring* if the right (resp. left) annihilator of an element of R is generated by an idempotent. Also, a ring R is called a *p.p.-ring* if it is both right and left p.p.-ring.

In [3], the following fact was proved.

PROPOSITION 2.1 *The following statements are equivalent:*

(1) *R is a right p.q.-Baer ring.*

(2) *The right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*

(3) *The right annihilator of every principal right ideal is generated (as a right ideal) by an idempotent.*

(4) *The right annihilator of every finitely generated ideal is generated (as a right ideal) by an idempotent.*

Note that this statement is true if "right" is replaced by "left" throughout.

PROOF See [3].

In [3], they also have shown the following results:

THEOREM A. *R is a right (resp. left) p.q.-Baer ring if and only if the polynomial ring $R[x]$ is a right (resp. left) p.q.- Baer ring.*

THEOREM B. *For a ring R, the following statements are equivalent:*

(1) *R is a quasi-Baer ring;*

(2) *the polynomial ring $R[x]$ over R is a quasi-Baer ring;*

(3) *the formal power series ring $R[[x]]$ over R is a quasi-Baer ring.*

Now we try to apply those results for the skew polynomial rings. Also, we recalled that R is an *abelian* ring if every idempotent of R

is central. We can observe easily that every reduced ring is abelian and in a reduced ring R left and right annihilaters coincide for any subset U of R, where a left (right) annihilator of U is denoted by $l_R(U) = \{a \in R \mid aU = 0\}(r_R(U) = \{a \in R \mid Ua = 0\})$.

Of course, for a reduced ring R, the following statements are equivalent clearly:

(1) R is a right p.p.-ring.

(2) R is a left p.p.-ring.

(3) R is a right p.q.-Baer ring.

(4) R is a left p.q.-Baer ring.

According to Theorem 1 1 in Section 1, if R is an integral domain with a monomorphism α, then the skew polynomial ring $R[x, \alpha]$ is an integral domain and so a Baer ring. Also, according to Theorem 1.2 in Section 1, if R is a domain with α an inner automorphism of a ring R induced by an invertible element c (i.e. $\alpha(r) = c^{-1}rc$ for all $r \in R$), then the skew polynomial ring $R[x; \alpha]$ is a Baer ring by Corollary 2.7[3].

Now, for a quasi-Baer (or p.q.-Baer) ring, we have the following questions.

QUESTION 2 2

(1) If R is a quasi-Baer (or right (left) p.q.-Baer) ring, then is the Ore extension $R[x; \alpha, \delta]$ quasi-Baer (or right (left) p.q.-Baer)? Here α is an endomorphism of R and δ is α-derivation of R.

(2) Is the converse of (1) true?

EXAMPLE 2 3 (1) [2, Example 11] The ring $R = Z_2[x]/(x^2)$ is not quasi-Baer, where Z_2 is the field of two elements and (x^2) is the ideal of the ring $Z_2[x^2]$ generated by x^2. In fact, $l_R(R(x+(x^2))$ is not generated by an idempotent of R.

But since $R[y; \delta] \simeq Mat_2(Z_2[y^2])$, where a derivation δ is defined by $\delta(x+(x^2)) = 1+(x^2)$, $R[y; \delta]$ is quasi-Baer because $Z_2[y^2]$ is quasi-Baer and so $Mat_2(Z_2[y^2])$ is also quasi-Baer.

(2) [5, p 18] Let F be a field and $R = F[t]$ a polynomial ring over F with the endomorphism α given by $\alpha(f(t)) = f(0)$ for all $f(t) \in R$. Then R is a principal ideal domain but the skew polynomial ring $R[x; \alpha]$ is not an integral domain because $xt = \alpha(t)x = 0$. We will show that

the skew polynomial ring $R[x;\alpha]$ is neither a right p.q.-Baer nor a right p.p.-ring.

Consider a right ideal $xR[x;\alpha]$. Then

$$x\{f_0(t)+f_1(t)x+\cdots+f_n(t)x^n\}=f_0(0)x+f_1(0)x^2+\cdots+f_n(0)x^{n+1}$$

for all $f_0(t)+f_1(t)x+\cdots+f_n(t)x^n \in R[x;\alpha]$ and hence $xR[x;\alpha] = \{a_1x+a_2x^2+\cdots+a_nx^n | n\in N\cup\{0\},\ a_i\in F(i=0,1,...,n)\}$.

Note that $R[x;\alpha]$ has only idempotents 0 and 1 by simple computation.

Since $(a_1x+a_2x^2+\cdots+a_nx^n)1=(a_1x+a_2x^2+\cdots+a_nx^n)\neq 0$ for some nonzero element $a_1x+a_2x^2+\cdots+a_nx^n \in xR[x;\alpha]$, we get $1\notin r_{R[x;\alpha]}(xR[x;\alpha])$ and so $r_{R[x,\alpha]}(xR[x;\alpha])\neq R[x;\alpha]$.

Also, since $(a_1x+a_2x^2+\cdots+a_nx^n)t=0$ for all $a_1x+a_2x^2+\cdots+a_nx^n\in xR[x;\alpha]$, $t\in r_{R[x,\alpha]}(xR[x;\alpha])$ and hence $r_{R[x;\alpha]}(xR[x;\alpha])\neq 0$. Thus $r_{R[x;\alpha]}(xR[x;\alpha])$ is not generated by an idempotent. Therefore $R[x;\alpha]$ is not right p.q.-Baer and so neither quasi-Baer nor Baer.

Similarly, we can verify that $R[x;\alpha]$ is not a right p.p.-ring.

By Example 2.3, Question 2.2 above is not true and so we can ask " under what conditions, is Question 2.2 true ?". In [1], Armendariz proved that if R is a reduced ring, then R is a p.p. (resp. Baer)-ring if and only if the polynomial ring $R[x]$ is a p.p. (resp. Baer)-ring. We will generalize this result by showing that if R is a reduced ring with a monomorpism α of R and $\alpha(P)\subseteq P$ for any minimal prime ideal P in R, then R is a p.p. (resp. Baer)-ring if and only if the skew polynomial ring $R[x;\alpha]$ is a p.p. (resp. Baer)-ring. Based on these facts, we have the following:

COROLLARY 2 4 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P)\subseteq P$ for any minimal prime ideal P in R. If $f\in R[x;\alpha]$ is an idempotent, then $f\in R$, that is, every idempotent of $R[x;\alpha]$ is an idempotent of R.*

PROOF. Let $f=a_0+a_1x+\cdots+a_nx^n\in R[x;\alpha]$ be an idempotent. Then $0=f-f^2=f(1-f)$. By Proposition 1.7 in Section 1, $a_0(1-a_0)=0$ and $a_i^2=0$ for each i $(1\leq i\leq n)$, and we get $a_0=a_0^2$ and so $a_i=0$ for each i $(1\leq i\leq n)$. Hence $f=a_0\in R$.

By the same method, we have

COROLLARY 2.5 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. If $f \in R[[x;\alpha]]$ is an idempotent, then $f \in R$, that is, every idempotent of $R[[x;\alpha]]$ is an idempotent of R.*

COROLLARY 2.6. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. If $T \subseteq R[X;\alpha]$ and $S_f = \{a_0, a_1, \ldots, a_n\}$, where $f = a_0 + a_1 x + \cdots + a_n x^n \in T$, then $r_{R[x,\alpha]}(T) = r_R(S_T)[x;\alpha]$, where $S_T = \cup_{f \in T} S_f$.*

PROOF If $g = b_0 + b_1 x + \cdots + b_m x^m \in r_{R[x,\alpha]}(T)$, then $Tg = 0$, i.e., $fg = 0$ for all $f \in T$ By Proposition 1.7 in Section 1, $a_i b_j = 0$ for all i and j ($0 \leq i \leq m$, $0 \leq j \leq n$), which implies that $b_j \in r_R(S_T)$, and so $g \in r_R(S_T)[x;\alpha]$. Hence $r_{R[x,\alpha]}(T) \subseteq r_R(S_T)[x;\alpha]$. The other inclusion is obvious.

Similarly, we have

COROLLARY 2.7 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. If $T \subseteq R[[X;\alpha]]$ and $S_f = \{a_0, a_1, \ldots\}$, where $f = \sum_{i=0}^{\infty} a_i x^i \in T$, then $r_{R[[x,\alpha]]}(T) = r_R(S_T)[[x;\alpha]]$, where $S_T = \cup_{f \in T} S_f$.*

THEOREM 2.8 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a p.p.-ring if and only if R is a p.p.-ring.*

PROOF ($\Longrightarrow$) If $R[x;\alpha]$ is a p.p.-ring and $a \in R$, then $r_R(a) = R \cap r_{R[x,\alpha]}(a) = R \cap eR[x;\alpha]$ for some idempotent $e \in R[x;\alpha]$. By Corollary 2.4, $e \in R$, and so $r_R(a) = eR$. Hence R is a p.p.-ring.

($\Longleftarrow$) Assume that R is a p.p. ring. Note that for any finite subset T of R, $r_R(T) = eR$ for some idempotent $e \in R$. If $f \in R[x;\alpha]$, then by Corollary 2.6, $r_{R[x;\alpha]}(f) = r_R(S_f)[x;\alpha] = eR[x;\alpha]$ for some idempotent $e \in R$ because S_f is a finite subset of R and e is central. Hence $R[x;\delta]$ is a p.p.-ring.

Since a p.p.-ring is equivalent to a p q.-Baer ring for a reduced ring, we have the following:

COROLLARY 2.9. *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a p.q.-Baer ring if and only if R is a p.q.-Baer ring.*

Similarly, we can also have

THEOREM 2.10 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a Baer ring if and only if R is a Baer ring.*

PROOF. ($\Longrightarrow$) If $R[x;\alpha]$ is Baer, then for any subset T of R, $r_{R[x;\alpha]}(T) = fR[x;\alpha]$ for some idempotent $f \in R[x;\alpha]$. By Corollary 2.4, $f \in R$, and then $r_R(T) = R \cap r_{R[x;\alpha]}(T) = R \cap fR[x;\alpha] = fR$. Hence R is a Baer ring.

($\Longleftarrow$) Suppose that R is Baer and T is an arbitrary subset of $R[x;\delta]$. Let $S_T = \cup_{f \in T} S_f$. Since R is Baer, $r_R(S_T) = eR$ for some idempotent $e \in R$. By Corollary 2.6, $r_{R[x,\alpha]}(T) = r_R(S_T)[x;\alpha] = eR[x;\alpha]$. Thus $R[x;\alpha]$ is Baer.

THEOREM 2.11 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[[x;\alpha]]$ is a Baer ring if and only if R is a Baer ring.*

PROOF. It can be proved by the similar method in the proof of Theorem 2.10 and using Corollary 2.7.

Theorems 2.8 and 2.10 extend Armendariz's results[1, Theorem A and B] if α is the identity. Also, for a reduced ring R, the following are equivalent clearly:

(1) R is a Baer ring.

(2) R is a quasi-Baer ring.

Hence we have

COROLLARY 2.12 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[x;\alpha]$ is a quasi-Baer ring if and only if R is a quasi-Baer ring.*

PROOF. It follows from the above fact and Corollary 1.9.

COROLLARY 2.13 *Let R be a reduced ring with a monomorpism α. Assume that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R. Then $R[[x;\alpha]]$ is a quasi-Baer ring if and only if R is a quasi-Baer ring.*

PROOF. It follows from the above fact and Corollary 1.10.

All results in this paper does not hold if the endomorphism α of a reduced ring R is not a monomorphism even though $\alpha(P) \subseteq P$ for any minimal prime ideal P in R.

For an example, let F be a field and $R = F[[t]]$ the formal power series ring over F with the endomorphism α given by $\alpha(f(t)) = f(0)$ for all $f(t) \in R$. In this case, $R = F[[t]]$ is a domain and so R is a Baer ring and also (0) is a unique minimal prime ideal. Since $\alpha(0) = (0)$, the assumption that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R is satisfied. But we will show that the skew polynomial ring $R[x;\alpha]$ is not a p.p.- ring

Consider a right ideal $xR[x;\alpha]$. Then

$$x\{f_0(t) + f_1(t)x + \cdots + f_n(t)x^n\} = f_0(0)x + f_1(0)x^2 + \cdots + f_n(0)x^{n+1}$$

for all $f_0(t) + f_1(t)x + \cdots + f_n(t)x^n \in R[x;\alpha]$ and hence $xR[x;\alpha] = \{a_1x + a_2x^2 + \cdots + a_nx^n \mid n \in N \cup \{0\}, a_i \in F(i = 0, 1, ..., n)\}$.

Note that $R[x;\alpha]$ has only idempotents 0 and 1 by simple computation.

Since $(a_1x + a_2x^2 + \cdots + a_nx^n)1 = (a_1x + a_2x^2 + \cdots + a_nx^n) \neq 0$ for some nonzero element $a_1x + a_2x^2 + \cdots + a_nx^n \in xR[x;\alpha]$, we get $1 \notin r_{R[x,\alpha]}(xR[x;\alpha])$ and so $r_{R[x,\alpha]}(xR[x;\alpha]) \neq R[x;\alpha]$.

Also, since $(a_1x + a_2x^2 + \cdots + a_nx^n)t = 0$ for all $a_1x + a_2x^2 + \cdots + a_nx^n \in xR[x;\alpha]$, we get $t \in r_{R[x,\alpha]}(xR[x;\alpha])$ and hence $r_{R[x,\alpha]}(xR[x;\alpha]) \neq 0$. Thus $r_{R[x;\alpha]}(xR[x;\alpha])$ is not generated by an idempotent. Therefore $R[x;\alpha]$ is not right p.q.-Baer and so neither quasi-Baer nor Baer.

Similarly, we can verify that $R[x;\alpha]$ is not a right p.p.-ring.

We finish this paper with raising the following question.

QUESTION 2 15 (1) Let R be a reduced ring with an automorphism α. Then $R[x;\alpha]$ is a p.q.(resp. quasi)-Baer ring if and only if R is a p.q.(resp. quasi)-Baer ring.

(2) For an abelian ring R with the monomorphism α satisfying that $\alpha(P) \subseteq P$ for any minimal prime ideal P in R, are the results in this paper true?

Acknowledgements

The author thanks Dr. Jae Keol Park at Pusan National University, Dr. Yasuyuki Hirano at Okayama University and Dr. Gary F. Birkenmeier at USL for their kind comments and helpful suggestions.

REFERENCES

[1] E P Armendariz, *A note on extensions of Baer and p.p. rings*, Australian Math Soc **18** (1974), 470 – 473
[2] E P Armendariz, H K. Koo and J K. Park, *Isomorphic Ore extensions*, Comm in Algebra **15(12)** (1987), 2633 – 2652
[3] G F Birkenmeier, J.Y Kim and J.K. Park, *On extensions of quasi-Baer and principally quasi-Baer rings, preprint*
[4] K R Goodearl and R.B Warfield, JR., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge 1989
[5] J C McConnell and J C. Robson, *Noncommutative Noetherian rings*, John Wiley & Sons Ltd 1987.

Department of Mathematics
Gyeongsang National University
Jinju 660-701, Korea
E-mail: hkkim@gshp.gsnu.ac.kr