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ITERATIVE SOLUTIONS TO NONLINEAR EQUATIONS OF THE ACCRETIVE TYPE IN BANACH SPACES

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ABSTRACT In this paper, we prove that under certain conditions the Ishikawa iterative method with errors converges strongly to the unique solution of the nonlinear strongly accretive operator equation $Tx = f$. Related results deal with the solution of the equation $x + Tx = f$ Our results extend and improve the corresponding results of Liu, Chidume, Chidume-Osilike, Tan-Xu, Deng, Deng-Ding and others.

1. Introduction and Preliminaries

Let X be a real Banach space and denote its norm and dual by $\|\cdot\|$ and X^*, respectively. A nonlinear operator T with domain $D(T)$ and range $R(T)$ in X is said to be *accretive* (Browder [1] and Kato [11]) if for all $x, y \in D(T)$ and $r > 0$, there holds the inequality

$$\|x - y\| \leq \|x - y + r(Tx - Ty)\|. \tag{1.1}$$

T is accretive if and only if for any $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y)\rangle \geq 0, \tag{1.2}$$

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where

$$Jx = \{f \in X^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad x \in X,$$

is the normalized duality mapping of X and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X and X^*. If T is accretive and $(1 + rT)(D(T)) = X$ for all $r > 0$, then T is called *m-accretive*. If for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ and a constant $k \in (0, 1)$ such that

$$\langle Tx - Ty, j(x - y)\rangle \geq k\|x - y\|^2, \tag{1.3}$$

then T is called *strongly accretive*. It is well known (see, for example, Theorem 13.1 of Deimling [7]) that for given $f \in X$, the equation

$$Tx = f \tag{1.4}$$

has a unique solution if $T : X \to X$ is strongly accretive and continuous. Martin [14] has also proved that if $T : X \to X$ is continuous and accretive, then T is m-accretive so that for given $f \in X$ the equation

$$x + Tx = f \tag{1.5}$$

has a unique solution.

Several authors have applied the Mann iterative method and the Ishikawa iterative method to approximate solutions of equations (1.4) and (1.5). (See, for example, [2]-[4], [8]-[11], [16]). The objective of this paper is to study the iterative approximation of solutions to the equation $Tx = f$ in the case when T is Lipschitzian and strongly accretive and X is an arbitrary real Banach space. Our results generalize most of the results that have appeared recently. In particular, the results of [2]-[5], [8]-[11], [13], [15], [16] and a host of others will be special cases of our theorems.

The following lemma plays a crucial role in the proofs of our main results.

LEMMA 1 1. ([13]) *Let $\{a_n\}_{n=0}^{\infty}$, $\{b_n\}_{n=0}^{\infty}$, $\{c_n\}_{n=0}^{\infty}$ be three nonnegative real sequences satisfying the inequality*

$$a_{n+1} \leq (1 - w_n)a_n + b_n w_n + c_n$$

for all $n \geq 0$, where $\{w_n\}_{n=0}^{\infty} \subset [0, 1]$, $\sum_{n=0}^{\infty} w_n = \infty$, $\lim_{n\to\infty} b_n = 0$ and $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

2. Main Results

In the sequel, $k \in (0,1)$ is the constant appearing in (1.3), l denotes the Lipschitz constant of T, $L = 1 + l$ and I stands for the identity operator on X.

THEOREM 2.1 *Let X be an arbitrary real Banach space, $T : X \to X$ be a Lipschitz strongly accretive operator and $f \in X$. Define the sequence iteratively by $x_0, u_0, v_0 \in X$,*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n S x_n + v_n, & n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n + u_n, & n \geq 0, \end{cases} \tag{2.1}$$

where $Sx = f + (I - T)x$ for all $x \in X$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ are two real sequences and $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ are two sequences in X satisfying the following conditions:

$$\sum_{n=0}^{\infty} \alpha_n = +\infty, \quad 0 \leq \alpha_n, \beta_n \leq 1, \quad n \geq 0, \tag{2.2}$$

$$\frac{k - L(L+1)\beta_n - L(L+1)(1+\beta_n l)\alpha_n}{1-(1-k)\alpha_n} \geq t, \quad n \geq 0, \tag{2.3}$$

$$\lim_{n\to\infty} \|v_n\| = 0, \quad \sum_{n=0}^{\infty} \|u_n\| < +\infty, \tag{2.4}$$

where $t \in (0,1)$ is a constant. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the solution of $Tx = f$.

PROOF. It follows from [1], [6] and the strong accretivity of T that the equation $Tx = f$ has a unique solution p in X. Then p is a fixed point of S and S is Lipschitz with constant L. It follows from (1.3) that for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-S)x - (I-S)y, j(x-y) \rangle \geq k\|x-y\|^2.$$

Thus

$$\langle (I-S-kI)x - (I-S-kI)y, j(x-y) \rangle \geq 0.$$

In view of (1.1) and (1.2), we have

$$\|x-y\| \le \|x-y+r[(I-S-kI)x-(I-S-kI)y]\| \tag{2.5}$$

for all $x, y \in X$ and $r > 0$. Using (2.1), we obtain that

$$\begin{aligned}(1-\alpha_n)x_n &= x_{n+1} - \alpha_n S y_n - u_n \\ &= [1-(1-k)\alpha_n]x_{n+1} + \alpha_n (I-S-kI)x_{n+1} \\ &\quad + \alpha_n S x_{n+1} - \alpha_n S y_n - u_n.\end{aligned} \tag{2.6}$$

Note that

$$(1-\alpha_n)p = [1-(1-k)\alpha_n]p + \alpha_n(I-S-kI)p. \tag{2.7}$$

It follows from (2.5)~(2.7) that

$$\begin{aligned}&(1-\alpha_n)\|x_n - p\| \\ &\ge [1-(1-k)\alpha_n]\|x_{n+1} - p + \frac{\alpha_n}{1-(1-k)\alpha_n}[(I-S-kI)x_{n+1} \\ &\quad - (I-S-kI)p]\| - \alpha_n\|Sx_{n+1} - Sy_n\| - \|u_n\| \\ &\ge [1-(1-k)\alpha_n]\|x_{n+1} - p\| - \alpha_n\|Sx_{n+1} - Sy_n\| - \|u_n\|,\end{aligned}$$

which implies that

$$\begin{aligned}&\|x_{n+1} - p\| \\ &\le \frac{1-\alpha_n}{1-(1-k)\alpha_n}\|x_n - p\| + \frac{\alpha_n}{1-(1-k)\alpha_n}\|Sx_{n+1} - Sy_n\| \\ &\quad + \frac{1}{1-(1-k)\alpha_n}\|u_n\|.\end{aligned} \tag{2.8}$$

We have the following estimates:

$$\begin{aligned}\|x_n - y_n\| &\le \beta_n\|x_n - Sx_n\| + \|v_n\| \\ &\le (L+1)\beta_n\|x_n - p\| + \|v_n\|,\end{aligned} \tag{2.9}$$

$$
\begin{aligned}
\|Sy_n - y_n\| &\leq (L+1)\|y_n - p\| \\
&\leq (L+1)(1-\beta_n + L\beta_n)\|x_n - p\| + (L+1)\|v_n\| \\
&\leq L(L+1)\|x_n - p\| + (L+1)\|v_n\|.
\end{aligned} \tag{2.10}
$$

By (2.1), (2.9) and (2.10), we yield that

$$
\begin{aligned}
&\|Sy_n - Sy_n\| \\
&\leq L\|x_{n+1} - y_n\| \\
&\leq L(1-\alpha_n)\|x_n - y_n\| + \alpha_n L\|Sy_n - y_n\| + L\|u_n\| \\
&\leq [L(L+1)\beta_n + L(L+1)(1+\beta_n l)\alpha_n]\|x_n - p\| \\
&\quad + L(L+1)\|v_n\| + L\|u_n\|.
\end{aligned} \tag{2.11}
$$

Using (2.11) in (2.8), we conclude that

$$
\begin{aligned}
&\|x_{n+1} - p\| \\
&\leq \left\{ \frac{1-\alpha_n}{1-(1-k)\alpha_n} + \frac{\alpha_n}{1-(1-k)\alpha_n}[L(L+1)\beta_n \right. \\
&\quad \left. + L(L+1)(1+\beta_n l)\alpha_n] \right\} \|x_n - p\| \\
&\quad + \frac{\alpha_n}{1-(1-k)\alpha_n} L(L+1)\|v_n\| + \frac{L}{1-(1-k)\alpha_n}\|u_n\| \\
&\leq \left[1 - \alpha_n \frac{k - L(L+1)\beta_n - L(L+1)(1+\beta_n l)\alpha_n}{1-(1-k)\alpha_n}\right]\|x_n - p\| \\
&\quad + D\alpha_n\|v_n\| + D\|u_n\|,
\end{aligned} \tag{2.12}
$$

where $D = \frac{L^2+L}{k}$. It follows from (2.3) and (2.12) that

$$
\|x_{n+1} - p\| \leq (1 - t\alpha_n)\|x_n - p\| + D\alpha_n\|v_n\| + D\|u_n\|.
$$

Put

$$
a_n = \|x_n - p\|, \quad w_n = t\alpha_n, \quad b_n = \frac{D}{t}\|v_n\| \quad \text{and} \quad c_n = D\|u_n\|
$$

for any $n \geq 0$. Then Lemma 1.1 ensures that $\|x_n - p\| \to 0$ as $n \to \infty$. This completes the proof.

THEOREM 2 2 *Let* X, f, T, $\{x_n\}_{n=0}^{\infty}$, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$, *be as in Theorem 2.1. Suppose that there exists a nonnegative sequence* $\{\gamma_n\}_{n=0}^{\infty}$ *with* $\lim_{n\to\infty}\gamma_n = 0$ *and* $\|u_n\| = \gamma_n\alpha_n$ *for any* $n \geq 0$. *Then* $\{x_n\}_{n=0}^{\infty}$ *converges strongly to the solution of* $Tx = f$.

PROOF Just as in the proof of Theorem 2.1, we have

$$\begin{aligned}\|x_{n+1} - p\| &\leq (1 - t\alpha_n)\|x_n - p\| + D\alpha_n\|v_n\| + D\|u_n\| \\ &= (1 - t\alpha_n)\|x_n - p\| + D\alpha_n(\|v_n\| + \gamma_n).\end{aligned}$$

Put

$$a_n = \|x_n - p\|, \quad w_n = t\alpha_n, \quad b_n = \frac{D}{t}(\|v_n\| + \gamma_n) \quad \text{and} \quad c_n = 0$$

for any $n \geq 0$. Then Lemma 1.1 ensures that $\|x_n - p\| \to 0$ as $n \to \infty$ completing the proof.

REMARK 2 1. Theorem 2.1 and Theorem 2.2 extend Theorem 1 of Liu [13], Theorem 1 of Childume [2], Theorem 2 of Childume [3], Theorems 1 and 3 of Childume and Osilike [4], Theorems 3.1 and 4.1 of Tan and Xu [16], Theorem 1 of Deng [8], [10], Theorems 1 and 3 of Deng [9] and Theorem 2 of Deng and Ding [11] from Banach spaces which are either uniformly convex or uniformly smooth to arbitrary real Banach spaces.

REMARK 2 2 The following example reveals that Theorem 2.1 extends properly Theorem 1 of Osilike [15].

EXAMPLE 2 1. Let X, f, T be as in Theorem 2.1 and

$$\begin{aligned}&t = \frac{k}{2}, \quad \alpha_n = \frac{k}{4L(L+1)+L}, \quad \beta_n = \frac{k}{4L(L+1)}, \\ &\|u_n\| = \frac{1}{(n+1)^2}, \quad \|v_n\| = \frac{1}{n+1}\end{aligned}$$

for all $n \geq 0$. Then the conditions of Theorem 2.1 are satisfied. But Theorem 1 in [15] is not applicable since $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ do not converge to 0.

THEOREM 2 3 *Let X be an arbitrary real Banach space and $T : X \to X$ be a Lipschitz accretive operator. Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.1, $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ satisfy (2.3) and*

$$(2.13) \qquad 1 - l(l+1)\beta_n - l(l+1)(1+\beta_n l)\alpha_n \geq t, \quad n \geq 0,$$

where $t \in (0,1)$ is a constant. Then for any given $f \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated from arbitrary $x_0, u_0, v_0 \in X$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n(f - Tx_n) + v_n, & n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n(f - Ty_n) + u_n, & n \geq 0 \end{cases}$$

converges strongly to the solution of $x + Tx = f$.

PROOF. It follows from Martin [14] and the accretivity of T that the equation $x + Tx = f$ has a unique solution $p \in X$. Define $S : X \to X$ by $Sx = f - Tx$. Then p is a fixed point of S and S is Lipschitz with the same Lipschitz constant as T. Furthermore, for all $x, y \in X$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle (I-S)x - (I-S)y, j(x-y) \rangle \geq \|x-y\|^2.$$

The rest of the argument is essentially the same as in the proof of Theorem 2.1 and is therefore omitted.

THEOREM 2 4 *Let X be an arbitrary real Banach space and $T : X \to X$ be a Lipschitz accretive operator. Let $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$ be as in Theorem 2.2 and $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ be as in Theorem 2.3. Then for any given $f \in X$, the sequence $\{x_n\}_{n=0}^{\infty}$ generated as in Theorem 2.3 converges strongly to the solution of the equation $x + Tx = f$.*

REMARK 2 3 Theorem 2.3 and Theorem 2.4 extend Corollary 9 of Chidume and Osilike [5] from Ishikawa iteration to Ishikawa iteration with errors, and improve Corollary 5 of Osilike [15]. The following example shows that Theorem 2.3 generalizes properly Corollary 5 in [15].

EXAMPLE 2 2 Let X, f, T be as in Theorem 2.1 and

$$t = \frac{1}{2}, \quad \alpha_n = \frac{1}{4l(l+1)+l}, \quad \beta_n = \frac{1}{4l(l+1)},$$
$$\|u_n\| = \frac{1}{(n+1)^2}, \quad \|v_n\| = \frac{1}{n+1}$$

for all $n \geq 0$. Then the conditions of Theorem 2.3 are satisfied. But Corollary 5 in [15] does not hold since $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ do not converge to 0.

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