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ON INTEGRAL REPRESENTATION WITH RESPECT TO VECTOR-VALUED FINITELY ADDITIVE MEASURES

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1. Introduction

In [5], [8], the authors considered the integral representation of bounded linear operators of $C(S, X)$ into Y, where S denotes a compact Hausdorff space, X and Y are Banach spaces, and $C(S, X)$ denotes the Banach space of all X-valued continuous functions defined on S.

It is well-known that an integration theory is to define the integral of a simple function and then extend the integral by some limit process to a general case of functions in [3], [4].

In [5], A. De Korvin and L. Kunes generated the integration theory of scalar-valued functions with respect to operator-valued measures obtained by D. R. Lewis in [6].

The purpose of this paper is to give an integral representation for the case of vector-valued functions with respect to finitely additive measure taking values in locally convex topological vector spaces, using both a weak and a strong approach.

2. Notations and preliminaries

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Let X be a normed space and Y be a locally convex Hausdorff linear topological space generated by the family Q of continuous seminorms on Y. Let X' and Y' be the topological duals of X and Y, respectively. Let $(S,\sum)$ be a measurable space and an operator-valued measure $\mu : \sum \to L(X,Y)$ be an additive set function with

$$\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n), \quad E_n \in \sum \text{ with } \cup_{n=1}^{\infty} E_n \in \sum,$$

$E_i \cap E_j = \emptyset (i \neq j), \ i, \ j = 1, 2, \cdots$, the series being unconditionally convergent with respect to the topology of simple convergence. Let us suppose that there exists a vector measure $\nu : \sum \to X$ and let μ be a non-negative real-valued measure on $\sum$. If $\lim_{\mu(E)\to 0}\nu(E) = 0$, then ν is called μ-continuous and this is denoted by $\nu \ll \mu$. When $\nu \ll \mu$, μ is sometimes said to be a control measure for ν.

It is well-known in [1] that if $\mu : \sum \to L(X,Y)$ is an operator-measure, then the set function $\mu_x : \sum \to Y$, defined by $\mu_x(E) = \mu(E)x$ is a vector measure and conversely, if $\mu(\cdot)x$ is a vector measure, then $\mu : \sum \to L(X,Y)$ is countably additive with respect to the topology of simple convergence in $L(X,Y)$. From the above result it can be proved that the set function $y'\mu : \sum \to X'$ defined by $(y'\mu)(E)x = y'(\mu(E)x)$ for $E \in \sum$ is an X'-valued measure. If $y' \in Y'$ and $q \in Q$, we will write $y' \leq q$ whenever $|y'(y)| \leq q(y)$ for $y \in Y$.

DEFINITION 2 1 ([3]) We define the q-variation of μ, which is a finitely set funtion on $\sum$, as

$$|\mu|_q(E) = \sup \sum_{i=1}^{n} q(\mu(E \cap E_i)), \quad E \in \sum,$$

where the supremum is taken over all finite pairwise disjoint sets $E_n \in \sum$. For each $y' \in Y'$, we write the variation of $y'\mu$, $|y'\mu|_q(\cdot)$, as

$$|y'\mu|_q(E) = \sup \sum_{i=1}^{n} \| y'\mu(E \cap E_i) \| .$$

DEFINITION 2.2 ([3], [4]). We define the q-semi-variation of μ as

$$\| \widehat{\mu} \|_q (E) = \sup_{y' \leq q} |y'\mu|_q(E), \quad E \in \sum,$$

which is non-negative. Note that $\| \widehat{\mu} \|_q (E) < 2 \sup_{F \subset E} \| y'\mu(F) \|$ and $\| \widehat{\mu} \|_q (E) < \infty$ whenever $\| y'\mu \|_q (E) < \infty$ for each $y' \in Y'$. It is proved easily that $\| \widehat{\mu} \|_q (\cdot)$ is monotone, subadditive and that $|\mu|_q(E) \leq \| \widehat{\mu} \|_q (E) \leq 4 \sup_{y' \leq q} \sup_{F \subset E} |y'\mu(F)|$.

We now develop an integration with respect to an operator-valued measure. Recall that in [1], [5] a sequence of functions (f_n) converges to f in semivariation if for every $\epsilon > 0, \delta > 0$ there exists some N such that for $n \geq N$, $\| \widehat{\mu} \|_q (\{s : |f_n - f| > \delta\}) < \epsilon$.

3. The weak integrals

Let $\mu : \sum \to L(X, Y)$ be a strongly finite measure with $|y'\mu|_q(E) < \infty$ for $E \in \sum$ and $y' \in Y'$. Also the integrands are assumed to be measurable.

DEFINITION 3 1 A function $f : S \to X$ is said to be a weakly μ-integrable if the following conditions hold.

(1) f is $y'\mu$ -integrable in the sense of [3].

(2) for $E \in \sum$ there exists an element $y_E \in Y$ such that $y'(y_E) = \int_E f dy'\mu$ for every $y' \in Y'$.

If f is μ-integrable, we denote $y_E = \int_E f d\mu$. We write sometimes $\int_E f d\mu$ for $\int_E f(s) d\mu(s)$. It follows from Definition 3.1 that every simple function with representation $f = \sum_{i=1}^{n} x_i \chi_{E_i} : S \to X$, where χ_{E_i} is the characteristic function of the sets $E_i \in \sum$, $x_i \in X$ and $E_i \cap E_j = \emptyset$ for $i \neq j$, $i, j = 1, 2, \cdots, n$, is μ-integrable over E and we define the integral of f as

$$\int_E f d\mu = \sum_{i=1}^{n} x_i \mu(E \cap E_i) \in X, \quad E \in \sum.$$

It is easily verified that if $f : S \to X$ is a simple function such that $q(f) = \sup q(f(s))$ for every $q \in Q$, $s \in S$, then $q(\int_E f d\mu) \leq \| f \|_s \| \widehat{\mu} \|_q (E)$ for $E \in \sum$, where $\| f \|_s = \sup_{s \in S} |f(s)|$.

LEMMA 3.2 *If $f : S \to X$ is $y'\mu$-integrable, then*
(1) $q(\int_E f d(y'\mu)) \le \int_E |f| d|y'\mu|$ for $E \in \sum$.
(2) the set function defined by $\phi(E) = \int_E f d\mu$ is a measure on $\sum$.

PROOF. (1) If f is $y'\mu$-integrable, then $|f|$ is $|y'\mu(\cdot)|$-integrable. Thus, if (f_n) is a defining sequence of simple functions for $y'\mu$-integrability of f, then $|f_n|$ is a defining sequence corresponding to the function $|f|$ and

$$q\left(\int_E f_n d(y'\mu)\right) \le \int_E |f_n| d|y'\mu| \quad \text{for } E \in \sum,$$

which completes the proof.

(2) From [3], p122, Proposition 5, since

$$\sum_{i=1}^{\infty} \int_{E_i} f d(y'\mu) = \int_{\bigcup_{i=1}^{\infty}} f d(y'\mu),$$

it follows

$$\sum_{i=1}^{\infty} y'\mu(E_i) = y'\mu(\bigcup_{i=1}^{\infty} E_i).$$

Hence μ is countably additive by Pettis theorem.

THEOREM 3.3. *Suppose Y is sequentially complete. If there is a sequence (f_n) of simple functions which converges to f on S, and $\int_E f d|y'\mu| = \lim_{n\to\infty} \int_E f_n d|y'\mu|$, then it follows*

$$\int_E f d\mu = \lim_{k\to\infty} \int_E f_{n_k} d\mu$$

for a subsequence (f_{n_k}) of (f_n).

PROOF Since a sequence (f_n) of simple functions converges to f, f_n is a bounded measurable function and $|f_n| \leq |f|$ for $n = 1, 2, \cdots$. For each $\epsilon > 0$, let $F_n = \{s \in S \cdot |f - f_n| > \epsilon\}$ and $E_n = \bigcup_{k=n}^{\infty} F_k$. Then, for every $q \in Q$,

$$
\begin{aligned}
q&\left(\int_E f_n d\mu - \int_E f_m d\mu\right) \\
&\leq \sup_{y' \leq q} \int_{E \cap E_n} |f - f_n| d|y'\mu| + \sup_{y' \leq q} \int_{E \cap E_n^c} |f| d|y'\mu| \\
&+ \sup_{y' \leq q} \int_{E \cap E_n^c} |f_n| d|y'\mu| + \sup_{y' \leq q} \int_{E \cap E_m} |f - f_m| d|y'\mu| \\
&+ \sup_{y' \leq q} \int_{E \cap E_m^c} |f| d|y'\mu| + \sup_{y' \leq q} \int_{E \cap E_m^c} |f_m| d|y'\mu| \\
&\leq \epsilon \parallel \widehat{\mu} \parallel_q (E \cap E_n^c) + 2 \parallel \widehat{\mu} \parallel_q (E \cap E_n) \\
&+ \epsilon \parallel \widehat{\mu} \parallel_q (E \cap E_m^c) + 2 \parallel \widehat{\mu} \parallel_q (E \cap E_m)
\end{aligned}
$$

for $E \in \sum$, which shows that $(\int_E f_n d(y'\mu))$ is Cauchy uniformly with respect to $E \in \sum$ and since Y is sequentially complete, there is an element y_E in Y such that $y'(y_E) = y'(\lim_{n\to\infty} \int_E f_n d\mu) = \int_E f d(y'\mu)$. Hence f is μ-integrable.

Let f be $|y'\mu|$-integrable, that is, there exists a sequence (f_n) of simple functions such that $\lim_{n\to\infty} \int_E |f_n - f| d|y'\mu| = 0$. Then

$$
\begin{aligned}
\sup_{y' \leq q} \int_E |f| d|y'\mu| &\leq \sup_{y' \leq q} \int_E |f - f_n| d|y'\mu| + \sup_{y' \leq q} \int_E |f_n||d(y'\mu)| \\
&\leq \infty ,
\end{aligned}
$$

which implies that f is μ-integrable. For each k there exists a n_k such that

$$
\sup_{y' \leq q} \int_E |f - f_{n_k}| d|y'\mu| < \frac{1}{k}.
$$

Let $x_{k,E} = \int_E f_{n_k} d\mu$ for $E \in \sum$. Then for $q \in Q$,

$$q(x_{n,E} - x_{m,E}) = q\left(\int_E (f_{n_k} - f_{m_k}) d\mu\right)$$
$$= \sup_{y' \leq q} |y' \int_E (f_{n_k} - f_{m_k}) d\mu|$$
$$\leq \sup_{y' \leq q} \int_E |f_{n_k} - f_{m_k}| d|y'\mu|.$$

Thus $x_{k,E}$ is Cauchy uniformly in Y, and therefore it converges. Since Y is sequentially complete, there exists an element $y_E \in Y$ such that $y_E = \lim_{k\to\infty} x_{k,E}$.

Then we show that for every $y' \in Y'$, $y'(y_E) = \int_E f d(y'\mu)$. In fact

$$|y'(x_{k,E}) - \int_E f d(y'\mu)| = \left|\int_E (f_{n_k} - f) d(y'\mu)\right|$$
$$\leq \sup_{y' \leq q} \int_E |f_{n_k} - f| d|y'\mu|$$
$$\to 0 \quad \text{as } k \to \infty,$$

and since $y'(y_E) = \lim_{k\to\infty} y'(x_{k,E})$, the assertion is proved. Hence f is μ-integrable and we have proven that

$$\int_E f d\mu = \lim_{k\to\infty} \int_E f_{n_k} d\mu \quad \text{for } E \in \sum.$$

THEOREM 3 4 *(1) Let (f_n) be a sequence of μ-integrable function which converges to f a.e on S with respect to μ,*

(2) let $|f_n| \leq g$ for each n and $g : S \to X$ be a $y'\mu$-integrable function,

(3) for every $\epsilon > 0$ there exists $\delta > 0$ such that $\| \widehat{\mu} \|_q (E) < \delta$ implies

$$\left|\int_E g d(y'\mu)\right| < \epsilon \quad \text{for } y' \in \sigma'.$$

Then f is μ-integrable whenever Y is sequentially complete and $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$ uniformly for $E \in \sum$.

PROOF We first show that $(\int_E f_n d\mu)$ is Cauchy uniformly with respect to $E \in \sum$. For given $\epsilon > 0$, let $\phi(E) = \int_E g d\mu$, $F_n = \{s \in S : |f - f_n| > \epsilon\}$ and $E_n = \cup_{k=n}^{\infty} F_k$. Then (E_n) is a decreasing sequence of set $E_n \searrow \emptyset$.

By applying the dominated convergence theorem for operator-valued measure, f is $y'\mu$-integrable and for each n, we see that f is $y'\mu$-integrable and $\int_E f d(y'\mu) = \lim_{n\to\infty} \int_E f_n d(y'\mu)$.

Now for $q \in Q$,

$$\begin{aligned}
&\left|\int_E (f - f_n) d(y'\mu)\right| \\
&\quad \le \sup_{y' \le q} \left|\int_{E \cap E_n^c} (f - f_n) d(y'\mu)\right| + \sup_{y' \le q} \left|\int_{E \cap E_n} (f - f_n) d(y'\mu)\right| \\
&\quad \le \epsilon \parallel \widehat{\mu} \parallel_q (E \cap E_n^c) + 2 \sup_{y' \le q} \int_{E \cap E_n} \parallel g \parallel d|y'\mu| \\
&\quad = \epsilon \parallel \widehat{\mu} \parallel_q (S) + 2 \parallel \phi \parallel_q (E_n).
\end{aligned}$$

Thus,

$$\begin{aligned}
&q\left(\int_E f_n d\mu - \int_E f_m d\mu\right) \\
&\quad \le 2\epsilon \parallel \widehat{\mu} \parallel_q (S) + 2 \sup_{y' \le q} \int_{E \cap E_n} |g| d|y'\mu| + 2 \sup_{y' \le q} \int_{E \cap E_m} |g| d|y'\mu| \\
&\quad < 2\epsilon \parallel \widehat{\mu} \parallel_q (S) + 2 \parallel \phi \parallel_q (E_n) + 2 \parallel \phi \parallel_q (E_m),
\end{aligned}$$

for all n, m and $E \in \sum$. So the sequence $(\int_E f_n d\mu)$ is Cauchy uniformly with respect to $E \in \sum$. Since Y is sequentially complete, there is an element y_E in Y such that $y'(y_E) = y'(\lim_{n\to\infty} \int_E f_n d\mu) = \int_E f d(y'\mu)$ for $E \in \sum$. Hence f is μ-integrable and $\lim_{n\to\infty} \int_E f_n d\mu = \int_E f d\mu$.

DEFINITION 3.5 ([2]). A function $f : S \to X$ is said to be μ-integrable if there exists an X-valued sequence (f_n) of simple functions such that

(1) $f_n \to f$ μ-a.e.,

(2) given $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\parallel \mu \parallel_q (E) < \delta$ for $E \in \sum$ implies $q(\int_E f_n d\mu) < \epsilon$ for all $n \in N$.

THEOREM 3.6. *Let $\| \widehat{\mu} \|_q$ be finite and $f_n \to f$ μ-a.e.. Let $g : S \to X$ be $y'\mu$-integrable such that $|f_n| \le g$, $n = 1, 2, \cdots$, $|f_n - g| \le M$, $|f - g| \le M$ for some constant M. If f is $y'\mu$-integrable, then $\int_E f_n d(y'\mu)$ converges to $\int_E f d(y'\mu)$ uniformly for $y' \in Y'$, $E \in \sum$. If Y is sequentially complete, then f is μ-integrable.*

PROOF. Let $E_n = \cup_{n=1}^{\infty} \left\{ s \in S : |f_n - f| \ge \frac{\epsilon}{4\|\widehat{\mu}\|_q(S)} \right\}$. Then (E_n) is a decreasing sequence of sets with $E_n \searrow \emptyset$ and so, given $\epsilon > 0$, there exits a positive integer $N = N(\epsilon)$ such that $\| \widehat{\mu} \|_q (E_n) < \frac{\epsilon}{8M}$ for all $n \ge N$ and all $s \in S \cap E_n$ and $q(f_n - f) \to 0$ uniformly on $S \cap E_n^c$. Then

$$\begin{aligned}\left| \int_E (f - f_n) d(y'\mu) \right| &\le \left| \int_{E \cap E_n^c} (f - f_n) d(y'\mu) \right| + \left| \int_{E \cap E_n} (f - f_n) d(y'\mu) \right| \\ &\le \frac{\epsilon}{4 \| \widehat{\mu} \|_q (S)} (E \cap E_n^c) + 2M \| \widehat{\mu} \|_q (E_n) \\ &< \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}.\end{aligned}$$

Since $\| \widehat{\mu} \|_q (E_n) \to 0$ as $n \to \infty$, we see that $q(\int_E (f - f_n) d(y'\mu))$ converges to 0 uniformly for $y' \in Y'$ and $E \in \sum$.

Thus now for given $\epsilon > 0$,

$$\begin{aligned}&q\Big(\int_E f_n d(y'\mu) - \int_E f_m d(y'\mu) \Big) \\ &\qquad < \epsilon(\| \widehat{\mu} \|_q (E \cap E_n^c) + 2M) + \epsilon(\| \widehat{\mu} \|_q (E \cap E_m^c) + 2M),\end{aligned}$$

for all $n, m \ge N$ and $E \in \sum$.

So the sequence $(\int_E f_n d\mu)$ is Cauchy uniformly with respect to $E \in \sum$.

By applying the dominated convergence theorem, we see that f is $y'\mu$-integrable and

$$\lim_{n \to \infty} \int_E f_n d(y'\mu) = \int_E f d(y'\mu) \quad \text{for } E \in \sum.$$

Since Y is sequentially complete, there exists an element y_E in Y such that $y'(y_E) = y'(\lim_{n\to\infty} \int_E f_n d\mu) = \int_E f d(y'\mu)$. Hence f is μ-integrable and $\int_E f d\mu = \lim_{n\to\infty} \int_E f_n d\mu$.

4. The strong integrals

In this section we shall assume that μ is strongly bounded, that is, for every sequence of pairwise disjoint sets (A_n) in $\sum$, $\lim_{n\to\infty} \mu(A_n) = 0$. Then every $\| \mu \|_q$ is strongly bounded for $q \in Q$.

To prove this, note that

$$\begin{aligned}\| \mu \|_q (A) &= \sup_{y' \le q} |y'\mu|(A) \\ &= \sup_{y' \le q} \{ \sup_{B \subset A} (y'\mu(B) + y'\mu(A - B))\} \\ &= \sup_{B \subset A} \sup_{y' \le q} \{y'\mu(B) + y'\mu(A - B)\} \\ &\le \sup_{B \subset A} \{q(\mu(B)) + q(\mu(A - B))\} \\ &\le 2 \sup_{B \subset A} q(\mu(B)).\end{aligned}$$

If (A_n) is a sequence of pairwise disjoint sets in $\sum$, and assume, by contradiction, that there exists $q' \in Q$ such that $\| \mu \|_{q'} (A_n) \not\to 0$, then there exists $\epsilon_0 > 0$ such that for all k there is an $n_k > k$, $k \in N$ with $\sup_{A_{n_k} \subset A_n} q'\mu(A_{n_k}) > \epsilon_0$; thus, for every k there exists $B_{n_k} \subset A_{n_k}$ such that $q'(\mu(B_{n_k})) > \sup_{A_{n_k} \subset A_n} q'(\mu(A_{n_k})) - \frac{\epsilon_0}{2} > \frac{\epsilon_0}{2}$.

Since the A_{n_k}'s are pairwise disjoint, so are the B_{n_k}'s, but cannot be $\mu(B_{n_k}) \to 0$, which is contradiction.

DEFINITION 4.1. For any simple function $f = \sum_{i=1}^{n} x_i \chi_{E_i}$ and for $E \in \sum$ we define $\int_E f d\mu = \sum_{i=1}^{n} x_i \mu(E \cap E_i)$; a function $f : S \to X$ is said to be strongly μ-integrable if there exists a sequence (f_n) of simple functions such that

(1) for every $\epsilon > 0$ and $q \in Q$, $\| \widehat{\mu} \|_q (\{|f - f_n| > \epsilon\}) \to 0$,

(2) the sequence $(\int_E f_n d\mu)$ converges in Y.

Then we put

$$(s)-\int_E f d\mu = \lim_{n\to\infty}\int_E f_n d\mu,\quad E\in\sum \tag{*}$$

Note that, trivially, if f is simple, the μ-integrable and the strong μ-integrable coincide.

THEOREM 4 2. *Let (f_n) and (g_n) be defining sequences for f such that they both satisfy (1) and (2) of Definition 4.1, then*

(1) $\lim_{n\to\infty}\int_E f_n d\mu = \lim_{n\to\infty}\int_E g_n d\mu$

(2) $\lim_{n\to\infty}\int_E f_n d\mu = (s)-\int_E f d\mu$ is uniform with respect to $E\in\sum$.

(3) $(s)-\int_S f d\mu \ll \mu$ in the sense of the $\|\cdot\|_q$-variation for every $q\in Q$.

PROOF. Let $h_n = g_n - f_n$. It is evident that (h_n) converges to 0 in $\|\cdot\|_q$-variation for every $q\in Q$. Also there exists $y_E = \lim_{n\to\infty}\int_E h_n d\mu$ for $E\in\sum$. Let $\epsilon>0$ be fixed, and define $A_n = \{s\in S : |h_n| > \epsilon\}$ and $B_n = \cup_{n=k}^{\infty} A_k$. So for $y'\in Y'$ there exists a positive integer n_0 such that $|y'\mu|(B_n) < \epsilon$ for all $n\geq n_0$. Since h_n's are simple, they are bounded, say $|h_n|\leq M_n$, where $M_n = \sup_{s\in S}|h_n(s)|$; Then for $E\in\sum$,

$$\begin{aligned} q\Big(\int_E h_n d\mu\Big) &\leq q\Big(\int_{E\cap A_n^c} h_n d\mu\Big) + q\Big(\int_{E\cap A_n} h_n d\mu\Big)\\ &= \sup_{y'\leq q}\int_{E\cap A_n^c} |h_n| d|y'\mu| + \sup_{y'\leq q}\int_{E\cap A_n} |h_n| d|y'\mu|\\ &\leq \epsilon \sup_{y'\leq q} |y'\mu|(E\cap A_n^c) + M_n|y'\mu|(E\cap A_n)\\ &< \epsilon(\|\mu\|_q\,(E\cap A_n^c) + M_n)\quad \text{for } E\in\sum. \end{aligned}$$

By Vitali-Hahn-Saks Theorem [3], the $\int_E h_n d\mu$ are $\|\mu\|_q$-continuous uniformly with respect to n and $\lim_{n\to\infty}\int_{E\cap E_n} f_n d\mu = 0$ for $E\in\sum$. Thus $q(y_E)=0$; since Y is Hausdorff, and $q\in Q$ is arbitrary, this yields $y_E = 0$. Moreover, in similar way, we can show that if the sequence (f_n) of simple functions converges to f, then for every $q \in Q$, $q(\int_E f_n d\mu) \ll \| \widehat{\mu} \|_q (E)$ uniformly with respect to n, that is, for every $q \in Q$, and $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \widehat{\mu} \|_q (E) < \delta$ implies that $q(\int_E f_n d\mu) < \epsilon$ for $n = 1, 2, \cdots$, and thus $q[(s) - \int_E f d\mu] < \epsilon$ for $q \in Q$, i.e.,

$$q\left((s) - \int_E f d\mu\right) \ll \| \widehat{\mu} \|_q (E) \quad \text{for } q \in Q,\ E \in \sum .$$

So, $q(\int_E f d\mu) \ll \| \mu \|_q (E)$ for $n = 1, 2, \cdots$ yields that the limit in (*) is uniform. In fact, since $\| \widehat{\mu} \|_q (\{|f_n - f| > \epsilon\}) \to 0$ as $n \to \infty$, there exists-a $\delta > 0$ such that $\| \widehat{\mu} \|_q (\{|f_n - f| > \epsilon\} \cap E) < \delta$ for $E \in \sum$, and hence $q[(s) - \int_{\{|f - f_n| > \epsilon\} \cap E} (f_n - f)] < \epsilon$ for $n = 1, 2, \cdots$.

To prove (3), since

$$\begin{aligned}
q\left((s) - \int_{\{|f_n - f| > \epsilon\} \cap E} (f_n - f) d\mu\right) \\
&\leq \sup_{y' \leq q} \left((s) - \int_S (f_n - f) d\mu\right)(\{|f_n - f| > \epsilon\}) \\
&\leq \| (s) - \int_S (f_n - f) d\mu \|_q (\{|f_n - f| > \epsilon\}) \\
&\leq \epsilon \| \widehat{\mu} \|_q (S)
\end{aligned}$$

for $E \in \sum$, hence, for $n = 1, 2, \cdots$ and $E \in \sum$,

$$\begin{aligned}
q\left((s) - \int_E (f_n - f) d\mu\right) &\leq q\left((s) - \int_{\{|f_n - f| > \epsilon\} \cap E} (f_n - f) d\mu\right) \\
&\quad + q\left((s) - \int_{\{|f_n - f| > \epsilon\}^c \cap E} (f_n - f) d\mu\right) \\
&< \epsilon + \epsilon \| \widehat{\mu} \|_q (S).
\end{aligned}$$

Finally, since $(s) - \int_{(\ )} f d\mu = \lim_{n \to \infty} \int_{(\ )} f_n d\mu$, and $q(\int_E f_n d\mu)$ is $\| \mu \|_q$-continuous, uniformly with respect to n, we find that given $\epsilon > 0$ there exists a $\delta > 0$ such that $\| \widehat{\mu} \|_q (E) < \delta$ yields $q(\int_E f_n d\mu) < \frac{\epsilon}{2}$, $n = 1, 2, \cdots$ and $q((s) - \int_E f d\mu) < \frac{\epsilon}{2}$; thus if $\sup_{F \subset E} q((s) - \int_F f d\mu) < \frac{\epsilon}{2}$, then $\| (s) - \int_E f d\mu \|_q < \epsilon$, and this completes the proof.

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