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SOME RESULTS ON GENERALIZED LIE IDEALS WITH DERIVATION

NEŞET AYDIN, KÂZIM KAYA AND OZNUR GOLBAŞI

ABSTRACT Let R be a prime ring with characteristic not two U a (σ,τ)-left Lie ideal of R and d $R \to R$ a non-zero derivation The purpose of this paper is to invesitigate identities satisfied on prime rings We prove the following results (1) $[d(R), a] = 0 \Leftrightarrow d([R, a]) = 0$. (2) if $(R, a)_{\sigma,\tau} = 0$ then $a \in Z$ (3) if $(R, a)_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $a \in Z$. (4) if $(U, a) \subset Z$ then $a^2 \in Z$ or $\sigma(u) + \tau(u) \in Z$, for all $u \in U$. (5) if $(U, R)_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $U \subset Z$.

1. Introduction

Let R be a ring and σ, τ be two mappings from R into itself. We write $[x, y]$, (x, y), $[x, y]_{\sigma,\tau}$, $(x, y)_{\sigma,\tau}$ for $xy - yx$, $xy + yx$, $x\sigma(y) - \tau(y)x$ and $x\sigma(y) + \tau(y)x$ respectively and make extensive use of basic commutator identities. $(xy, z) = x[y, z] + (x, z)y = x(y, z) - [x, z]y$, $[xy, z]_{\sigma,\tau} = x[y, z]_{\sigma,\tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma,\tau}y$. We set $C_{\sigma,\tau} = \{c \in R \mid c\sigma(x) = \tau(x)c$, for all $x \in R\}$ and call (σ, τ)-*center* of R.

An additive mapping $D : R \to R$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is *inner* if there exits an $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$.

For subsets $A, B \subset R$, let $[A, B]$ ($[A, B]_{\sigma,\tau}$) be the additive subgroup generated by all $[a, b]$ ($[a, b]_{\sigma,\tau}$) for all $a \in A$ and $b \in B$. We recall that

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a *Lie ideal*, L is an additive subgroup of R such that $[R, L] \subset L$. We first introduce the generalized Lie ideal in [6] as following. Let U be an additive subgroup of R, $\sigma, \tau : R \to R$ two mappings. Then (i) U is a $(\sigma, \tau)-$right Lie ideal of R if $[U, R]_{\sigma,\tau} \subset U$. (ii) U is a $(\sigma, \tau)-$left Lie ideal of R if $[R, U]_{\sigma,\tau} \subset U$. (iii) U is both a $(\sigma, \tau) - right$ Lie ideal and $(\sigma, \tau)-$left Lie ideal of R then U is a $(\sigma, \tau)-$Lie ideal of R. Every Lie ideal of R is a $(1, 1)-$left Lie ideal of R, where $1 : R \to R$ is the identity map. As an example, let I be the set of integers,

$$R = \left\{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \mid x, y, z, t \in I \right\},$$

$$U = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in I \right\} \subset R,$$

and $\sigma, \tau : R \to R$ the mappings defined by $\tau(x) = axa$, $\sigma(x) = bxb^{-1}$, where $a = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \in R$. Then U is a $(\sigma, \tau)-$left Lie ideal but not a Lie ideal of R. Some algebraic properties of $(\sigma, \tau)-$Lie ideals are considered in [2], [3] and [7], where further references can be found.

Some authors have proved commutativity theorems for prime rings which are centralizing on appropriate subset of R. One would naturally wonder if some of these theorems are peculiar to Lie commutator. Our main objective is to investigate connections either between Jordan commutator and a $(\sigma, \tau)-$center or between Jordan commutator and generalized Lie ideals of prime ring.

2. Results

LEMMA 1. *[5, Lemma 3] Let R be a prime ring and $a, b \in R$. If $a, ab \in C_{\sigma,\tau}$ then $a = 0$ or $b \in Z$.*

LEMMA 2 *Let R be a prime ring, $d : R \to R$ a derivation and $a \in R$. $[d(R), a] = 0$ if and only if $d([R, a]) = 0$.*

PROOF. By [4, Theorem] $a \in Z$. Hence $[R, a] = 0$, and so $d([R, a]) = 0$. Conversely, replace r by ar, $r \in R$, we get $0 = d([ar, a]) = d(a[r, a]) =$

$d(a)[r,a] + ad([r,a])$. By hypothesis,

$$d(a)[r,a] = 0, \qquad \forall r \in R. \tag{2.1}$$

In (2.1), replace r by rs, $s \in R$ and use (2.1), we have $0 = d(a)[rs,a] = d(a)r[s,a] + d(a)[r,a]s$ and so we have

$$d(a)R[s,a] = 0 \qquad \forall s \in R.$$

Since R is a prime ring we obtain

$$d(a) = 0 \quad \text{or} \quad a \in Z.$$

If $d(a) = 0$ then we have $0 = d([r,a]) = [d(r),a] + [r,d(a)]$. It gives us $[d(R),a] = 0$. On the other hand if $a \in Z$ then we also have in the former case, $[d(R),a] = 0$.

COROLLARY 1 *If $d([R,a]) = 0$ then $a \in Z$.*

PROOF By Lemma 2, $[d(R),a] = 0$ and by [4, Theorem], $a \in Z$.

COROLLARY 2 *If $d([R,R]) = 0$ then R is commutative.*

LEMMA 3 *If $(R,a)_{\sigma,\tau} = 0$ then $a \in Z$.*

PROOF If $x,y \in R$ then $0 = (xy,a)_{\sigma,\tau} = x[y,\sigma(a)] + (x,a)_{\sigma,\tau}y = x[y,\sigma(a)]$. Thus we have

$$R[R,\sigma(a)] = 0.$$

Since R is prime we have $a \in Z$.

COROLLARY 3. *Let R be a prime ring. If $(R,R)_{\sigma.\tau} = 0$ then R is commutative.*

PROOF. By Lemma 3, it is obvious.

THEOREM 1 *If $(R,a)_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $a \in Z$.*

PROOF By hypothesis, for all $x \in R$, $(x,a)_{\sigma,\tau} \in C_{\sigma,\tau}$. Replace x by $x\sigma(a)$, we obtain $(x\sigma(a),a)_{\sigma,\tau} = x[\sigma(a),\sigma(a)] + (x,a)_{\sigma,\tau}\sigma(a) = (x,a)_{\sigma,\tau}\sigma(a) \in C_{\sigma,\tau}$. By Lemma 1, we have

$$(R,a)_{\sigma,\tau} = 0 \quad \text{or} \quad a \in Z$$

by Lemma 3, we have $a \in Z$.

COROLLARY 4 *If $(R,a) \subset Z$ then $a \in Z$.*

COROLLARY 5. *If $(R,R) \subset Z$ then R is commutative.*

COROLLARY 6 *Let U be a $(\sigma,\tau)-$Lie ideal. If $(R,U)_{\sigma,\tau} \subset C_{\sigma,\tau}$ then $U \subset Z$.*

THEOREM 2. *Let U be a $(\sigma,\tau)-$left Lie ideal and $a \in R$. If $(U,a) \subset Z$ then $a^2 \in Z$ or for all $u \in U$, $\sigma(u) + \tau(u) \in Z$.*

PROOF. For all $u \in U$, $(u,a) = ua + au \in Z$. But for all $x \in R$ and all $u \in U$, $[x,u]_{\sigma,\tau} \in U$. We have $a[x,u]_{\sigma,\tau} + [x,u]_{\sigma,\tau}a \in Z$. Thus $0 = [a[x,u]_{\sigma,\tau} + [x,u]_{\sigma,\tau}a, a] = a[[x,u]_{\sigma,\tau},a] + [[x,u]_{\sigma,\tau},a]a = ([[x,u]_{\sigma,\tau},a],a)$, and so

$$([[x,u]_{\sigma,\tau},a],a) = 0 \text{ for all } x \in R, u \in U.$$

Now, if we consider the relation

$$([[x,u]_{\sigma,\tau},a],a) = [[x,u]_{\sigma,\tau},(a,a)] - ([[x,u]_{\sigma,\tau},a],a)$$

we have $2[[x,u]_{\sigma,\tau},a^2] = 0$. Since $char R \neq 2$ we obtain

$$[[x,u]_{\sigma,\tau},a^2] = 0 \text{ for all } x \in R, u \in U. \tag{2.2}$$

Taking $x\sigma(u)$ instead of x in (2.2) and using (2.2) we get

$$\begin{aligned} 0 &= [[x\sigma(u),u]_{\sigma,\tau},a^2] = [x[\sigma(u),\sigma(u)] + [x,u]_{\sigma,\tau}\sigma(u),a^2] \\ &= [[x,u]_{\sigma,\tau}\sigma(u),a^2] = [x,u]_{\sigma,\tau}[\sigma(u),a^2] + [[x,u]_{\sigma,\tau},a^2]\sigma(u) \end{aligned}$$

and so,

$$[x,u]_{\sigma,\tau}[\sigma(u),a^2]=0, \quad \text{for all} \quad x\in R, u\in U. \tag{2.3}$$

In (2.3), replace x by xy, $y\in R$ and use (2.3) we have

$$0=[xy,u]_{\sigma,\tau}[\sigma(u),a^2]=x[y,u]_{\sigma,\tau}[\sigma(u),a^2]+[x,\tau(u)]y[\sigma(u),a^2].$$

This implies that $[R,\tau(u)]R[\sigma(u),a^2]=0$. Sine R is a prime ring the last equation gives $u\in Z$ or $[\sigma(u),a^2]=0$ for all $u\in U$ and so, we have

$$[U,\sigma^{-1}(a^2)]=0.$$

By [1, Lemma 6], we get $a^2\in Z$ or $\sigma(u)+\tau(u)\in Z$ for all $u\in U$.

THEOREM 3 *Let R be a prime ring and U a $(\sigma,\tau)-$left Lie ideal of R. If $(U,R)_{\sigma,\tau}\subset C_{\sigma,\tau}$ then $U\subset Z$.*

PROOF. If R is commutative then U$\subset Z$. So we assume that R is not commutative. For any $u\in U$ and $r\in R$, $0=[(u,r)_{\sigma,\tau},r]_{\sigma,\tau}=[u\sigma(r)+\tau(r)u,r]_{\sigma,\tau}=u[\sigma(r),\sigma(r)]+[u,r]_{\sigma,\tau}\sigma(r)+\tau(r)[u,r]_{\sigma,\tau}+[\tau(r),\tau(r)]u=[u,r]_{\sigma,\tau}\sigma(r)+\tau(r)[u,r]_{\sigma,\tau}$. We obtain

$$([u,r]_{\sigma,\tau},r)_{\sigma,\tau}=0 \text{ for all } r\in R, u\in U. \tag{2.4}$$

Expanding (2.4) one obtains $0=([u,r]_{\sigma,\tau},r)_{\sigma,\tau}=(u\sigma(r)-\tau(r)u,r)_{\sigma,\tau}=u\sigma(r)\sigma(r)-\tau(r)u\sigma(r)+\tau(r)u\sigma(r)-\tau(r)\tau(r)u=u\sigma(r^2)-\tau(r^2)u$ and so,

$$[u,r^2]_{\sigma,\tau}=0 \qquad \textit{for all } r\in R, u\in U. \tag{2.5}$$

Substituting $r+s$ for r in (2.5) and using (2.5) we obtain $0=[u,(r+s)^2]_{\sigma,\tau}=[u,r^2+s^2+rs+sr]_{\sigma,\tau}=[u,r^2]_{\sigma,\tau}+[u,s^2]_{\sigma,\tau}+[u,rs]_{\sigma,\tau}+[u,sr]_{\sigma,\tau}=u\sigma(r)\sigma(s)-\tau(r)\tau(s)u+u\sigma(s)\sigma(r)-\tau(s)\tau(r)u=u(\sigma(r),\sigma(s))-(\tau(r),\tau(s))u$. This implies that

$$[u,(r,s)]_{\sigma,\tau}=0, \text{ for all } r,s\in R, u\in U. \tag{2.6}$$

Now, taking rs in place of r in (2.6) and using (2.6) one obtains $0 = [u,(rs,s)]_{\sigma,\tau} = [u, r[s,s]+(r,s)s]_{\sigma,\tau} = [u,(r,s)s]_{\sigma,\tau} = \tau((r,s))[u,s]_{\sigma,\tau}+ [u,(r,s)]_{\sigma,\tau}\sigma(s)$ and so, we have

$$(2.7) \qquad \tau((r,s))[u,s]_{\sigma,\tau} = 0, \text{ for all } r,s \in R, u \in U.$$

If we take rt, $t \in R$ instead of r in (2.7), it gives us $0 = \tau((rt,s))[u,s]_{\sigma,\tau} = \tau(r)\tau((t,s))[u,s]_{\sigma,\tau} - \tau([r,s])\tau(t)[u,s]_{\sigma,\tau} = \tau([s,r])\tau(t)[u,s]_{\sigma,\tau}$. Hence

$$\tau([s,R])\tau(R)[u,s]_{\sigma,\tau} = 0.$$

Since τ is an automorphism of R and R is a prime ring it implies that

$$s \in Z \text{ or } [U,s]_{\sigma,\tau} = 0.$$

We set $K = \{s \in R \mid s \in Z\}$ and $L = \{s \in R \mid [U,s]_{\sigma,\tau} = 0\}$. Clearly each of L and K is an additive subgroup of R. Moreover, R is the set-theoretic union of L and K. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, $R \subset Z$ which forces R to be commutative, i.e., $U \subset Z$. Therefore

$$(2.8) \qquad [U,s]_{\sigma,\tau} = 0, \text{ for all } s \in R.$$

From (2.8), $u\sigma(s) - \tau(s)u = 0$, i.e.

$$u\sigma(s) = \tau(s)u \qquad \textit{for all } s \in R, u \in U.$$

By hypothesis, for all $s,r \in R$, $u \in U$, $(u,rs)_{\sigma,\tau} \in C_{\sigma,\tau}$. Expanding this and using (2.8) one obtain $(u,rs)_{\sigma,\tau} = u\sigma(rs) - \tau(rs)u = u\sigma(r)\sigma(s) + \tau(r)u\sigma(s) = (u\sigma(r) + \tau(r)u)\sigma(s) = (u,r)_{\sigma,\tau}\sigma(u) \in C_{\sigma,\tau}$. Since $(u,r)_{\sigma,\tau} \in C_{\sigma,\tau}$ by Lemma 1 we have

$$(2.9) \qquad (U,r)_{\sigma,\tau} = 0, \text{ for all } r \in R.$$

Considering (2.8) together with (2.9), one obtains

$$2u\sigma(s) = 0, \text{ for all } s \in R, u \in U.$$

Since $charR \neq 2$, we get $U\sigma(R) = 0$. By [7, Lemma 1 (iii)] we have $U \subset Z$.

THEOREM 4 *Let R be a prime ring, $a \in R$ and U a $(\sigma,\tau)-$left Lie ideal of R. If $(U,a)_{\sigma,\tau}=0$ then $a \in Z$ or $\sigma(u)+\tau(u) \in Z$, for all $u \in U$.*

PROOF. By hypothesis, for any $r \in R$, $u \in U$,

$$\begin{aligned}0 &= ([ur,u]_{\sigma,\tau},a)_{\sigma,\tau} = (\tau(u)[r,u]_{\sigma,\tau}+[\tau(u),\tau(u)]\sigma(u),a)_{\sigma,\tau}\\ &= (\tau(u)[r,u]_{\sigma,\tau},a)_{\sigma,\tau} = \tau(u)([r,u]_{\sigma,\tau},a)_{\sigma,\tau} - [\tau(u),\tau(a)][r,u]_{\sigma,\tau}\\ &= [\tau(a),\tau(u)][r,u]_{\sigma,\tau}\end{aligned}$$

and so, we have

$$(2.10) \qquad [\tau(u),\tau(a)][r,u]_{\sigma,\tau}=0, \text{ for all } r \in R, u \in U.$$

Replacing r by rs, $s \in R$, in (2.10) and using (2.10) we have

$$\begin{aligned}0 &= [\tau(u),\tau(a)][rs,u]_{\sigma,\tau}\\ &= [\tau(u),\tau(a)][r,u]_{\sigma,\tau}s + [\tau(u),\tau(a)]r[s,u]_{\sigma,\tau}\\ &= [\tau(u),\tau(a)]r[s,u]_{\sigma,\tau}.\end{aligned}$$

That is,

$$[\tau(u),\tau(a)]R[R,u]_{\sigma,\tau}=0$$

Since R is prime, we get $[U,a]=0$. By [1, Lemma 6] we obtain $a \in Z$ or $\sigma(u)+\tau(u) \in Z$, for all $u \in U$.

THEOREM 5. *Let R be a prime ring and U a $(\sigma,\tau)-$left Lie ideal of R. If $(R,U)_{\sigma,\tau}=0$ then $U \subset Z$.*

PROOF For any $x,y \in R$ and $u \in U$, $0=(xy,u)_{\sigma,\tau}=x[y,\sigma(u)]+(x,u)_{\sigma,\tau}y = x[y,\sigma(u)]$. Since R is prime we get $U \subset Z$.

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Mersin University
Faculty of Arts and Science
Department of Mathematics
Mersin - TURKEY
E-mail: neseta@mersin.edu.tr
URL· http://www.mersın.edu.tr

18 Mart University
Faculty of Arts and Science
Department of Mathematics
Cąnakkale- TURKEY
E-mail: kkaya@comu.edu.tr
URL. http://www.comu.edu.tr

Cumhuriyet University
Faculty of Arts and Science
Department of Mathematics
Sivas - TURKEY
E-maıl: ogolbasi@cumhuriyet.edu.tr
URL http://www.cumhurıyet.edu.tr