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MODIFICATIONS OF PRODUCT CONVERGENCE STRUCTURES

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ABSTRACT In this paper, we introduce the notion of some modification of given convergence structure and product convergence. Also, we find some properties which hold between the modification associated with a product of convergence structures and the product of modifications associated with the factor convergence structures

I. Introduction

A convergence structure defined by Kent [4] is a correspondence between the filters on a given set X and the subsets of X which specifies which filters converge to which points of X. This concept is defined to include types of convergence which are more general than that defined by specifying a topology on X. Thus, a convergence structure may be regarded as a generalization of a topology.

With a given convergence structure q on a set X, Kent [4] introduced associated convergence structures which are called a topological modification, a pretopological modification and a pseudotopological modification. Also, Kent [2] introduced product convergence structures.

In this paper, we shall study some properties of the product convergence structure of given convergence structures and their modifications. In particular, we will show that the pseudotopological modification of

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the product convergence structure is equal to the product convergence structure of the pseudotopological modifications of factor convergence structures.

Finally, we obtain some inequalities which hold between modifications of the product convergence structure and the product convergence structure of modifications associated with the factor convergence structures.

II. Preliminaries

A *convergence structure* q on a set X is defined to be a function from the set $F(X)$ of all filters on X into the set $P(X)$ of all subsets of X, satisfying the following conditions:

(1) $x \in q(\dot{x})$ for all $x \in X$;

(2) $\Phi \subset \Psi$ implies $q(\Phi) \subset q(\Psi)$;

(3) $x \in q(\Phi)$ implies $x \in q(\Phi \cap \dot{x})$,

where $\dot{x}$ denotes the principal ultrafilter containing $\{x\}$; Φ and Ψ are in $F(X)$ Then the pair (X, q) is called a *convergence space*. If $x \in q(\Phi)$, then we say that Φ *q-converges* to x. The filter $V_q(x)$ obtained by intersecting all filters which q-converge to x is called the *q-neighborhood filter* at x. If $V_q(x)$ q-converges to x for each $x \in X$, then q is said to be *pretopological* and the pair (X, q) is called a *pretopological space*. A convergence structure q is said to be *pseudotopological* if Φ q-converges to x whenever each ultrafilter finer than Φ q-converges to x, and the pair (X, q) is called a *pseudotopological space*.

A convergence structure q is said to be *topological* if q is pretopological and for each $x \in X$, the filter $V_q(x)$ has a filter base $B_q(x)$ with the following property:

$$y \in G(x) \in B_q(x) \text{ implies } G(x) \in B_q(y).$$

Let $C(X)$ be the set of all convergence structures on X, partially orded as follows:

$$q_1 \leq q_2 \text{ iff } q_2(\Phi) \subset q_1(\Phi) \text{ for all } \Phi \in F(X).$$

If $q_1 \leq q_2$, then we say that q_1 is coarser than q_2, and q_2 is finer than q_1.

For any $q \in C(X)$, we define the following related convergence structures, $\rho(q), \pi(q)$ and $\lambda(q)$:

(1) $x \in \rho(q)(\Phi)$ iff $x \in q(\Phi')$ for each ultrafilter Φ' finer than Φ.

(2) $x \in \pi(q)(\Phi)$ iff $V_q(x) \subset \Phi$.

(3) $x \in \lambda(q)(\Phi)$ iff $U_q(x) \subset \Phi$, where $U_q(x)$ is the filter generated by the sets $U \in V_q(x)$ which have the property: $y \in U$ implies $U \in V_q(y)$. In this case, $\rho(q), \pi(q)$ and $\lambda(q)$ are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of q, and the pairs $(X, \rho(q)), (X, \pi(q))$ and $(X, \lambda(q))$ are called the *pseudotopological modification*, the *pretopological modification* and the *topological modification* of (X, q), respectively.

PROPOSITION 1([4]) *(1) $\rho(q)$ is the finest pseudotopology coarser than q.*

(2) $\pi(q)$ is the finest pretopology coarser than q.

(3) $\lambda(q)$ is the finest topology coarser than q.

(4) $\lambda(q) \leq \pi(q) \leq \rho(q) \leq q$.

Let f be a map from X into Y and Φ a filter on X. Then $f(\Phi)$ means the filter generated by $\{f(F) \mid F \in \Phi\}$.

Let f be a map from a convergence space (X, q) to a convergence space (Y, p). Then f is said to be *continuous* at a point $x \in X$, if the filter $f(\Phi)$ on Y p-converges to $f(x)$ for every filter Φ on X q-converging to x. If f is continuous at every point $x \in X$, then f is said to be continuous. Also, f is said to be *neighborhood preserving*, if $V_p(f(x)) = f(V_q(x))$.

PROPOSITION 2([6]) *If $f\colon (X, q) \to (Y, p)$ is continuous at $x \in X$, then $V_p(f(x)) \subset f(V_q(x))$.*

Let (X_λ, q_λ) be a convergence space, $X = \prod_{\lambda \in \Lambda} X_\lambda$ the product of sets and $P_\lambda\colon X \to (X_\lambda, q_\lambda)$ the λ-th projection for each $\lambda \in \Lambda$. The *product convergence structure* q on X is defined by specifying that for any $x \in X$ and $\Phi \in F(X)$,

$$x \in q(\Phi) \text{ iff } P_\lambda(x) \in q_\lambda(P_\lambda(\Phi)) \text{ for each } \lambda \in \Lambda.$$

In this case, the product convergence structure q is denoted by $\prod_{\lambda \in \Lambda} q_\lambda$ and the pair (X, q) is called the product convergence space of $\{(X_\lambda, q_\lambda) \mid$

$\lambda \in \Lambda\}$. The product convergence structure q is the coarsest convergence structure on X with respect to which all projections $P_\lambda: X \to (X_\lambda, q_\lambda)$ are continuous ([9]). We know that, given a filter Φ_λ on X_λ for each $\lambda \in \Lambda$, the family $\{P_\lambda^{-1}(B_\lambda) \mid B_\lambda \in \Phi_\lambda, \lambda \in \Lambda\}$ has the finite intersection property. The *product filter* of $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ means the filter on X which has a base the set of subsets of X of the form $\cap_{\lambda\in\Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in \Phi_\lambda$ for each $\lambda \in \Lambda$ and Λ' is a finite subset of Λ. The product filter of $\{\Phi_\lambda \mid \lambda \in \Lambda\}$ is denoted by $\prod_{\lambda\in\Lambda} \Phi_\lambda$ and this product filter $\Phi = \prod_{\lambda\in\Lambda} \Phi_\lambda$ is the coarsest filter on X such that $P_\lambda(\Phi) = \Phi_\lambda$ for each $\lambda \in \Lambda$(See p64, [1]).

The following Proposition 3 and Proposition 4 are immediate results of above definitions.

PROPOSITION 3 *Let (X_λ, q_λ) be a convergence space, Φ_λ a filter on X_λ and $P_\lambda: X = \prod_{\lambda\in\Lambda} X_\lambda \to X_\lambda$ the λ-th projection for each $\lambda \in \Lambda$. Then, for a filter Φ on X, $x \in X$ and $q = \prod_{\lambda\in\Lambda} q_\lambda$, the following hold:*

(1) $\Phi_\lambda = P_\lambda(\prod_{\lambda\in\Lambda} \Phi_\lambda)$. (2) $\prod_{\lambda\in\Lambda} P_\lambda(\Phi) \subset \Phi$.

(3) P_λ is neighborhood preserving.

(4) $\prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x)) \subset V_q(x)$(If q_λ is pretopological for each $\lambda \in \Lambda$, then the equality holds.).

PROPOSITION 4 *Let q_λ and p_λ be convergence structures on X_λ for each $\lambda \in \Lambda$.*

If $q_\lambda \le p_\lambda$ for each $\lambda \in \Lambda$, then, $\prod_{\lambda\in\Lambda} q_\lambda \le \prod_{\lambda\in\Lambda} p_\lambda$.

PROPOSITION 5 ([9]) *Let (X_λ, q_λ) be a convergence space for each $\lambda \in \Lambda$ and (X, q) the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. Then the following hold:*

q_λ is pretopological for each $\lambda \in \Lambda$ iff q is pretopological.

PROPOSITION 6. *Let X_λ be a nonempty set for each $\lambda \in \Lambda$. Let $\mathcal{F}$ be a filter on $X = \prod_{\lambda\in\Lambda} X_\lambda$ and $\mathcal{M}_\mu$ be an ultrafilter on X_μ with $\mathcal{M}_\mu \supset P_\mu(\mathcal{F})$. Then, there is an ultrafilter $\mathcal{M}$ on X such that $\mathcal{M} \supset \mathcal{F}$ and $P_\mu(\mathcal{M}) = \mathcal{M}_\mu$.*

PROOF Let $\Phi = \{A \times \prod_{\lambda\in\Lambda-\{\mu\}} X_\lambda \mid A \in \mathcal{M}_\mu\}$ and $\mathcal{B} = \{A \cap B \mid A \in \mathcal{F}, B \in \Phi\}$. Then, $\mathcal{B}$ is a filter base on X. Let Ψ be the filter generated by $\mathcal{B}$. There is an ultrafilter $\mathcal{M}$ on X such that $\Psi \subset \mathcal{M}$. Then $\mathcal{F} \subset \mathcal{M}$, and $\Phi \subset \mathcal{M}$.

On the other hand, let $A_\mu \in \mathcal{M}_\mu$ Then $A_\mu \times \prod_{\lambda\in\Lambda-\{\mu\}} X_\lambda) \in \Phi \subset \Psi \subset \mathcal{M}$, and so $A_\mu \in P_\mu(\mathcal{M})$. Therefore $\mathcal{M}_\mu \subset P_\mu(\mathcal{M})$. Thus, by the definition of ultrafilter, $\mathcal{M}_\mu = P_\mu(\mathcal{M})$. The proof is complete

III. Main Results

THEOREM 7 *Let (X_λ, q_λ) be a convergence space for each $\lambda \in \Lambda$ and (X, q) the product convergence space of $\{(X_\lambda, q_\lambda) \mid \lambda \in \Lambda\}$. Then the following hold:*

(1) If q_λ is topological for each $\lambda \in \Lambda$, then q is topological.

(2) q_λ is pseudotopological for each $\lambda \in \Lambda$ iff q is pseodotopological.

PROOF (1) Suppose that q_λ is topological for each $\lambda \in \Lambda$. Let $x \in X$ and $P_\lambda(x) = x_\lambda$. Then the filter $V_{q_\lambda}(x_\lambda)$ has a filter base $B_{q_\lambda}(x_\lambda)$ with the following property: $y_\lambda \in G_\lambda \in B_{q_\lambda}(x_\lambda)$ implies $G_\lambda \in B_{q_\lambda}(y_\lambda)$. Let $B_q(x)$ be the family of subsets of X of the form $\bigcap_{\lambda\in\Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in B_{q_\lambda}(x_\lambda)$ and Λ' is a finite subset of Λ. Then $B_q(x)$ is the filter base for the product filter $\prod_{\lambda\in\Lambda} V_{q_\lambda}(x_\lambda)$. Since q_λ is pretopological, $\prod_{\lambda\in\Lambda} V_{q_\lambda}(x_\lambda) = V_q(x)$. Therefore, $B_q(x)$ is the filter base for $V_q(x)$. Let $y \in G \in B_q(x)$ and $G = \bigcap_{\lambda\in\Lambda'} P_\lambda^{-1}(B_\lambda)$, where $B_\lambda \in B_{q_\lambda}(x_\lambda)$ and Λ' is a finite subset of Λ. Then $y \in P_\lambda^{-1}(B_\lambda)$ and hence $P_\lambda(y) \in B_\lambda$ for each $\lambda \in \Lambda'$. Thus, $B_\lambda \in B_{q_\lambda}(P_\lambda(y))$ and $G \in B_q(y)$. Consequently, q is topological.

(2) Suppose that q_λ is pseudotopological for each $\lambda \in \Lambda$. Let Φ be a filter on $X = \prod_{\lambda\in\Lambda} X_\lambda$. Then $x \in q(\mathcal{M})$ for all ultrafilters $\mathcal{M}$ finer than Φ on X. Then $P_\lambda(x) \in q_\lambda(P_\lambda(\mathcal{M}))$ for each $\lambda \in \Lambda$. Let $\mathcal{M}_\lambda$ be an ultrafilter on X_λ with $\mathcal{M}_\lambda \supset P_\lambda(\Phi)$. By Proposition 6, there is an ultrafilter $\mathcal{M}'$ on X such that $\mathcal{M}' \supset \Phi$ and $P_\lambda(\mathcal{M}') = \mathcal{M}_\lambda$. Thus, $P_\lambda(x) \in q_\lambda(P_\lambda(\mathcal{M}') = q_\lambda(\mathcal{M}_\lambda)$. Since q_λ is pseudotopological, we obtain $P_\lambda(x) \in q_\lambda(P_\lambda(\Phi))$ for each $\lambda \in \Lambda$. Thus, $x \in q(\Phi)$, and so q is pseudotopological.

Conversely, suppose that q is pseudotopological. Let Φ_λ be a filter on X_λ and $x_\lambda \in q_\lambda(\mathcal{M}_0)$ for all ultrafilters $\mathcal{M}_0$ finer than Φ_λ. Take a filter Ψ on X as follows ;

$$\Psi = \prod_{i\in\Lambda} \phi_i, \quad \phi_i = \begin{cases} \Phi_\lambda & \text{for } i = \lambda, \\ \dot{x}_i & \text{for } i \neq \lambda, \end{cases}$$

where $x_i \in X_i$.

Let $\mathcal{M}$ be an ultrafilter on X finer than Ψ. Then, $P_i(\mathcal{M}) \supset P_i(\Psi) = \phi_i$ for each $i \in \Lambda$. Thus, we obtain that $q_i(P_i(\mathcal{M})) \supset q_i(P_i(\Psi)) = q_i(\phi_i)$ and $x_i \in q_i(\phi_i)$ for each $i \in \Lambda - \{\lambda\}$. Also, $x_\lambda \in q_\lambda(P_\lambda(\mathcal{M}))$ since $P_\lambda(\mathcal{M})$ is an ultrafilter finer than Φ_λ. Therefore $x_i \in q_i(P_i(\mathcal{M}))$ for each $i \in \Lambda$. Thus, $x = (x_i)_{i\in\Lambda} \in q(\mathcal{M})$. Since q is pseudo-topological, we obtain $x = (x_i)_{i\in\Lambda} \in q(\Psi)$. Thus, $P_\lambda(x) = x_\lambda \in q_\lambda(P_\lambda(\Psi)) = q_\lambda(\Phi_\lambda)$. Consequently, q_λ is pseudotopological. The proof is complete.

The following Theorem 8 means that the pseudotopological modification of the product convergence structure is equal to the product convergence structure of the pseudotopological modifications of factor convergence structures

THEOREM 8 *Let $(X_\lambda, \rho(q_\lambda))$ be the pseudotopological modification of a convergence space $(X_\lambda,\ q_\lambda)$ for each $\lambda \in \Lambda$. Then $\rho(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \rho(q_\lambda)$.*

PROOF By Proposition 1, Proposition 4 and Theorem 7. it is clear that $\rho(\prod_{\lambda\in\Lambda} q_\lambda) \geq \prod_{\lambda\in\Lambda} \rho(q_\lambda)$. We show that $\rho(\prod_{\lambda\in\Lambda} q_\lambda) \leq \prod_{\lambda\in\Lambda} \rho(q_\lambda)$. Let $\mathcal{F}$ be a filter on $X = \prod_{\lambda\in\Lambda} X_\lambda$ and $x \in (\prod_{\lambda\in\Lambda} \rho(q_\lambda))(\mathcal{F})$. Then, $x_\lambda \in \rho(q_\lambda)(P_\lambda(\mathcal{F}))$ for each $\lambda \in \Lambda$. Let $\mathcal{M}$ be an ultrafilter finer than $\mathcal{F}$. Then $P_\lambda(\mathcal{M}) \supset P_\lambda(\mathcal{F})$, and so $P_\lambda(\mathcal{M})$ is an ultrafilter on X_λ. Thus $x_\lambda \in q_\lambda(P_\lambda(\mathcal{M}))$ for each $\lambda \in \Lambda$. Therefore, $x \in (\prod_{\lambda\in\Lambda} q_\lambda)(\mathcal{M})$ and $x \in (\rho(\prod_{\lambda\in\Lambda} q_\lambda)(\mathcal{F}))$.

Consequently, we obtain that $\prod_{\lambda\in\Lambda} \rho(q_\lambda)(\mathcal{F}) \subset (\rho(\prod_{\lambda\in\Lambda} q_\lambda))(\mathcal{F})$, that is, $\rho(\prod_{\lambda\in\Lambda} q_\lambda) \leq \prod_{\lambda\in\Lambda} \rho(q_\lambda)$. This proof is complete.

THEOREM 9. *Let $(X_\lambda, \pi(q_\lambda))$ be the pretopological modification of a convergence space (X_λ, q_λ) for each $\lambda \in \Lambda$. Then the following are equivalent:*

(a) $\prod_{\lambda\in\Lambda} \pi(q_\lambda) = \pi(\prod_{\lambda\in\Lambda} q_\lambda)$

(b) $V_{\prod_{\lambda\in\Lambda} q_\lambda}(x) = \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$ *for each* $x \in X = \prod_{\lambda\in\Lambda} X_\lambda$.

PROOF Let $q = \prod_{\lambda\in\Lambda} q_\lambda$ and $q^* = \prod_{\lambda\in\Lambda} \pi(q_\lambda)$.

(a) $\Longrightarrow$ (b): Suppose that $q^* = \pi(q)$, that is, $q^*(\Phi) = \pi(q)(\Phi)$ for every $\Phi \in F(X)$. Consider that $q^*(\Phi) = \{x \in X \mid P_\lambda(x) \in \pi(q_\lambda)(P_\lambda(\Phi))$ for each $\lambda \in \Lambda\} = \{x \in X \mid V_{q_\lambda}(P_\lambda(x)) \subset P_\lambda(\Phi)$ for each $\lambda \in \Lambda\}$ and $\pi(q)(\Phi) = \{x \in X \mid V_q(x) \subset \Phi\}$. Since $q^*(\Phi) = \pi(q)(\Phi)$, we obtain that

$$V_{q_\lambda}(P_\lambda(x)) \subset P_\lambda(\Phi) \text{ for each } \lambda \in \Lambda \text{ iff } V_q(x) \subset \Phi.$$

By Proposition 3, $V_{q_\lambda}(P_\lambda(x)) \subset P_\lambda(\prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$ for each $\lambda \in \Lambda$. Thus, $V_q(x) \subset \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$ and so $V_q(x) = \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$.

(b) $\Longrightarrow$ (a): By Proposition 1, Proposition 4 and Proposition 5, it is clear that $q^* \leq \pi(q)$. Suppose that $V_q(x) = \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(x))$ for each $\lambda \in \Lambda$ and $x \in X$. Let $\Phi \in F(X)$ and $y \in q^*(\Phi)$. Then $P_\lambda(y) \in \pi(q_\lambda)(P_\lambda(\Phi))$ and $V_{q_\lambda}(P_\lambda(y)) \subset P_\lambda(\Phi)$ for each $\lambda \in \Lambda$. Thus, $V_q(y) = \prod_{\lambda\in\Lambda} V_{q_\lambda}(P_\lambda(y)) \subset \prod_{\lambda\in\Lambda} P_\lambda(\Phi) \subset \Phi$ and $y \in \pi(q)(\Phi)$. Therefore, $q^*(\Phi) \subset \pi(q)(\Phi)$ and so $\pi(q) \leq q^*$. Consequently, $q^* = \pi(q)$.

Finally, we obtain the following Theorem 10

THEOREM 10 *Let $(X_\lambda, \lambda(q_\lambda)), (X_\lambda, \pi(q_\lambda))$ and $(X_\lambda, \rho(q_\lambda))$ be the topological, the pretopological and the pseudotopological modification of a convergence space (X_λ, q_λ) for each $\lambda \in \Lambda$, respectively. Then the following hold:*

$$(1) \prod_{\lambda\in\Lambda} \lambda(q_\lambda) \leq \lambda(\prod_{\lambda\in\Lambda} q_\lambda) \leq \pi(\prod_{\lambda\in\Lambda} q_\lambda).$$

$$(2) \prod_{\lambda\in\Lambda} \lambda(q_\lambda) \leq \prod_{\lambda\in\Lambda} \pi(q_\lambda) \leq \pi(\prod_{\lambda\in\Lambda} q_\lambda)$$

$$\leq \rho(\prod_{\lambda\in\Lambda} q_\lambda) = \prod_{\lambda\in\Lambda} \rho(q_\lambda) \leq \prod_{\lambda\in\Lambda} q_\lambda.$$

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