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NEW BOUNDS FOR A PERTURBED GENERALIZED TAYLOR'S FORMULA

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Abstract A generalised Taylor series with integral remainder involving a convex combination of the end points of the interval under consideration is investigated Perturbed generalised Taylor series are bounded in terms of Lebesgue p-norms on $[a,b]^2$ for $f_\Delta : [a,b]^2 \to \mathbb{R}$ with $f_\Delta(t,s) = f(t) - f(s)$.

1. Introduction

In the recent paper [1], Matić, Pečarić and Ujević proved the following generalised Taylor's formula.

Theorem 1 *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials, that is*

$$P_n'(t) = P_{n-1}(t),\ P_0(t) = 1,\ t \in \mathbb{R},\ n \in \mathbb{N},\ n \geq 1. \tag{1.1}$$

Further, let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. If $f : I \to \mathbb{R}$ is any function such that for some $n \in \mathbb{N}$, $f^{(n)}$ is absolutely continuous, then for any $x \in I$

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} [P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a)] + R_n(f; a, x), \tag{1.2}$$

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where

$$R_n(f;a,x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t)dt.$$

They also pointed out the following bounds for the remainder $R_n(f;\cdot,\cdot)$.

COROLLARY 1 *With the above assumptions, we have the estimations*

$$(1.3)\quad |R_n(f;a,x)| \leq \begin{cases} \|P_n\|_\infty \|f^{(n+1)}\|_1 & \textit{provided } f^{(n+1)} \in L_1[a,x], \\ \|P_n\|_q \|f^{(n+1)}\|_p & \textit{provided } f^{(n+1)} \in L_p[a,x],\ \ p>1,\ \frac{1}{p}+\frac{1}{q}=1, \\ \|P_n\|_1 \|f^{(n+1)}\|_\infty & \textit{provided } f^{(n+1)} \in L_\infty[a,x], \end{cases}$$

where $x \geq a$ and $\|\cdot\|_s$ $(1 \leq s \leq \infty)$ are the usual $s-$Lebesgue norms. That is,

$$\|g\|_s := \left(\int_a^x |g(t)|^s dt\right)^{\frac{1}{s}},\ s \in [1,\infty)$$

and

$$\|g\|_\infty := ess \sup_{t\in[a,x]} |g(t)|.$$

In this paper, the above results are evaluated for a specific polynomial $P_n(t)$ which involves a convex combination of the end points. Further, a perturbed generalised Taylor's formula is developed from which bounds are obtained in terms of Lebesgue $p-$norms on $[a,b]^2$ for $f_\Delta : [a,b]^2 \to \mathbb{R}$ with $f_\Delta(t,s) = f(t) - f(s)$. The results presented are an improvement on the pre-Grüss inequalities in [1] both in terms of the sharpness of the bounds and also in terms of the wider variety of norms. The results are applied to the polynomial involving a convex combination of the end points.

2. A Polynomial Involving a Convex Combination of the End Points

If we choose

$$\text{(P)}\qquad P_n(t) := \frac{1}{n!}[t - (\lambda a + (1-\lambda)x)]^n,$$

then, by (1.2), we obtain the representation

$$
\begin{aligned}
f(x) = \; & f(a) + \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-a)^k}{k!}[\lambda^k f^{(k)}(x) \\
& + (-1)^{k+1}(1-\lambda)^k f^{(k)}(a)] \\
& + \frac{(-1)^n}{n!}\int_a^x [t-(\lambda a+(1-\lambda)x)]^n f^{(n+1)}(t)dt,
\end{aligned} \tag{2.1}
$$

which generalises the following result from [1] (obtained for $\lambda = \frac{1}{2}$)
(2.2)

$$
\begin{aligned}
f(x) = \; & f(a) + \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-a)^k}{2^k k!}\left[f^{(k)}(x) + (-1)^{k+1} f^{(k)}(a)\right] \\
& + \frac{(-1)^n}{n!}\int_a^x \left(t - \frac{a+x}{2}\right)^n f^{(n+1)}(t)\,dt.
\end{aligned}
$$

If we apply Corollary 1 to the polynomial defined by (P), we can state the following theorem.

THEOREM 2 *Assume that f is as in Theorem 1. Then we have the inequality*

$$
\begin{aligned}
& \Bigg| f(x) - f(a) \\
& - \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-a)^k}{k!}\left[\lambda^k f^{(k)}(x) + (-1)^{k+1}(1-\lambda)^k f^{(k)}(a)\right]\Bigg|
\end{aligned} \tag{2.3}
$$

$$
\leq \begin{cases}
\frac{1}{n!}(x-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n \|f^{(n+1)}\|_1 \quad \text{if } f^{(n+1)} \in L_1[a,x]; \\
\frac{1}{n!(nq+1)^{\frac{1}{q}}}(x-a)^{n+\frac{1}{q}}\left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}} \|f^{(n+1)}\|_p \\
\text{if } f^{(n+1)} \in L_p[a,x],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\frac{1}{(n+1)!}(x-a)^{n+1}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right] \|f^{(n+1)}\|_\infty \\
\text{if } f^{(n+1)} \in L_\infty[a,x].
\end{cases}
$$

PROOF. We have

$$\begin{aligned}
\|P_n\|_\infty &= \frac{1}{n!}\sup_{t\in[a,x]}|t-(\lambda a+(1-\lambda)x)|^n \\
&= \frac{1}{n!}\left[\sup_{t\in[a,x]}|t-(\lambda a+(1-\lambda)x)|\right]^n \\
&= \frac{1}{n!}[\max\{\lambda a+(1-\lambda)x-a, x-(\lambda a+(1-\lambda)x)\}]^n \\
&= \frac{1}{n!}(x-a)^n[\max\{1-\lambda,\lambda\}]^n \\
&= \frac{1}{n!}(x-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n
\end{aligned}$$

and so

$$\|P_n\|_\infty = \frac{1}{n!}(x-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n \tag{2.4}$$

and the first inequality in (2.3) is proved.

In addition, as

$$\begin{aligned}
\|P_n\|_q &= \frac{1}{n!}\left[\int_a^x |t-(\lambda a+(1-\lambda)x)|^{nq}\,dt\right]^{\frac{1}{q}} \\
&= \frac{1}{n!}\left[\int_a^{\lambda a+(1-\lambda)x}(\lambda a+(1-\lambda)x-t)^{nq}\,dt\right. \\
&\quad \left.+\int_{\lambda a+(1-\lambda)x}^{x}(t-(\lambda a+(1-\lambda)x))^{nq}\,dt\right]^{\frac{1}{q}} \\
&= \frac{1}{n!}\left[\int_0^{(1-\lambda)(x-a)}u^{nq}du+\int_0^{\lambda(x-a)}v^{nq}dv\right]^{\frac{1}{q}}
\end{aligned}$$

and so

$$\|P_n\|_q = \frac{1}{n!}(x-a)^{n+\frac{1}{q}}\left[\frac{(1-\lambda)^{nq+1}+\lambda^{nq+1}}{nq+1}\right]^{\frac{1}{q}}, \tag{2.5}$$

the second inequality in (2.3) is also proved.

For $q = 1$, the assumptions are obvious and the theorem is thus proved.

Taking into account the fact that the mappings $f_1(\lambda) = \left[\frac{1}{2} + \left|\lambda - \frac{1}{2}\right|\right]^n$, $f_2(\lambda) = \left[(1-\lambda)^{nq+1} + \lambda^{nq+1}\right]^{\frac{1}{q}}$ and $f_3(\lambda) = (1-\lambda)^{n+1} + \lambda^{n+1}$ are convex and symmetrical about $\frac{1}{2}$, we may deduce that

$$\inf_{\lambda \in [0,1]} f_i(\lambda) = f_i\left(\frac{1}{2}\right), \quad i = 1, 2, 3$$

and then, the best inequality we can get from (2.3) is embodied in the following corollary.

COROLLARY 2. *Assume that f is as in Theorem 1. Then we have the inequality*

$$(2.6) \quad \left| f(x) - f(a) - \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{2^k k!} \left[f^{(k)}(x) + (-1)^{k+1} f^{(k)}(a) \right] \right|$$

$$\leq \begin{cases} \frac{1}{2^n n!} (x-a)^n \left\| f^{(n+1)} \right\|_1 ; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}} 2^n} (x-a)^{n+\frac{1}{q}} \left\| f^{(n+1)} \right\|_p \\ \text{if } f^{(n+1)} \in L_p[a,x],\ p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{(n+1)!2^n} (x-a)^{n+1} \left\| f^{(n+1)} \right\|_\infty \quad \text{if } f^{(n+1)} \in L_\infty[a,x]. \end{cases}$$

PROOF Taking $\lambda = \frac{1}{2}$ in (2.3) readily produces the stated result (2.6).

COROLLARY 3 *Let f have the properties stated in Theorem 2. Then the inequality*

$$\left| f(x) - f(a) - \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \right|$$

$$\leq \begin{cases} \frac{(x-a)^n}{n!} \left\| f^{(n+1)} \right\|_1, & f^{(n+1)} \in L_1[a,x]; \\ \frac{(x-a)^{n+\frac{1}{q}}}{n!(nq+1)^{\frac{1}{q}}} \left\| f^{(n+1)} \right\|_p, & f^{(n+1)} \in L_p[a,x], \\ & p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{(x-a)^{n+1}}{(n+1)!} \left\| f^{(n+1)} \right\|_\infty, & f^{(n+1)} \in L_\infty[a,x]. \end{cases}$$

holds.

PROOF Taking $\lambda = 0$ recaptures the classical Taylor series expansion above in terms of the $L_p[a,x]$, $p \geq 1$ Lebesgue norms (see for example, Dragomir [4]).

If we apply Theorem 2 for $f(x) = \int_a^x g(u)\,du$ and then choose $x = b$, we obtain the following trapezoid like inequality.

THEOREM 3 *Assume that the mapping $g : [a,b] \to \mathbb{R}$ is such that $g^{(n-1)}$ is absolutely continuous on $[a,b]$. Then we have the inequality*

$$\left| \int_a^b g(t)\,dt - \sum_{k=1}^{n} \frac{(-1)^{k+1}(x-a)^k}{k!} \right. \tag{2.7}$$
$$\left. \times \left[\lambda^k g^{(k+1)}(b) + (-1)^{k+1}(1-\lambda)^k g^{(k-1)}(a) \right] \right|$$

$$\leq \begin{cases} \frac{1}{n!}(b-a)^n\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^n \left\|g^{(n)}\right\|_1; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}}}(b-a)^{n+\frac{1}{q}}\left[(1-\lambda)^{nq+1}+\lambda^{nq+1}\right]^{\frac{1}{q}}\left\|g^{(n)}\right\|_p \\ \text{if } \ g^{(n)} \in L_p[a,b],\ p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(n+1)!}(b-a)^{n+1}\left[(1-\lambda)^{n+1}+\lambda^{n+1}\right]\|g^{(n)}\|_\infty \\ \text{if } \ g^{(n)} \in L_\infty[a,x]. \end{cases}$$

The following corollary contains the best inequality we may obtain from (2.7).

COROLLARY 4. *With the assumptions as in Theorem 3, we have the inequality:*

$$\left|\int_a^b g(t)\,dt - \sum_{k=1}^{n}(-1)^{k+1}\frac{(b-a)^k}{2^k k!}\left[g^{(k-1)}(b)+(-1)^{k+1}g^{(k-1)}(a)\right]\right| \tag{2.8}$$

$$\leq \begin{cases} \frac{1}{2^n n!}(b-a)^n\left\|g^{(n)}\right\|_1; \\ \frac{1}{n!(nq+1)^{\frac{1}{q}}2^n}(b-a)^{n+\frac{1}{q}}\left\|g^{(n)}\right\|_p & \text{if } \ g^{(n)} \in L_p[a,b], \\ & p>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \frac{1}{(n+1)!2^n}(b-a)^{n+1}\left\|g^{(n)}\right\|_\infty & \text{if } \ g^{(n)} \in L_\infty[a,x]. \end{cases}$$

REMARK 1 *The results in Theorem 3 and Corollary 4 can also be obtained from the 3 point quadrature formulae developed in the recent paper [3] by Cerone and Dragomir. We omit the details. Further, taking $\lambda = 0$ in Theorem 3 gives a representation of an integral in terms of a polynomial expansion about the end point a.*

3. Some Preliminary Results Involving Lebesgue Norms on $[a,b]^2$

For $f \in L_p[a,b]$ $(p \in [1,\infty))$ we can define the functional

$$\|f\|_p^{\triangle} := \left(\int_a^b \int_a^b |f(t)-f(s)|^p \, dtds\right)^{\frac{1}{p}} \tag{3.1}$$

and for $f \in L_\infty[a,b]$, we can define

$$\|f\|_\infty^{\triangle} := ess \sup_{(t,s)\in[a,b]^2} |f(t)-f(s)| \tag{3.2}$$

If we consider $f_\triangle : [a,b] \to \mathbb{R}$,

$$f_\triangle(t,s) = f(t) - f(s), \tag{3.3}$$

then, obviously

$$\|f\|_p^{\triangle} = \|f_\triangle\|_p, \quad p \in [1,\infty], \tag{3.4}$$

where, $\|\cdot\|_p$ are the usual Lebesque p-norms on $[a,b]^2$.

Using the properties of the Lebesque p-norms, we may deduce the following semi-norm properties for $\|\cdot\|_p^{\triangle}$:

(i) $\|f\|_p^{\triangle} \geq 0$ for $f \in L_p[a,b]$ and $\|f\|_p^{\triangle} = 0$ implies that $f = c$ (c is a constant) a.e. in $[a,b]$:

(ii) $\|f+g\|_p^{\triangle} \leq \|f\|_p^{\triangle} + \|g\|_p^{\triangle}$ if $f, g \in L_p[a,b]$:

(iii) $\|\alpha f\|_p^{\triangle} = |\alpha| \, \|f\|_p^{\triangle}$.

We note that if $p = 2$, then,

$$\begin{aligned}\|f\|_2^{\triangle} &= \left(\int_a^b \int_a^b f(t) - f(s)^2 \, dtds\right)^{\frac{1}{2}} \\ &= \sqrt{2}\left[(b-a)\|f\|_2^2 - \int_a^b f(t)\,dt)^2\right]^{\frac{1}{2}}\end{aligned}$$

If $f : [a,b] \to \mathbb{R}$ is absolutely continuous on $[a,b]$, then we can point out the following bounds for $\|f\|_p^{\triangle}$ in terms of $\|f'\|_p$.

THEOREM 4 *Assume that* $f : [a,b] \to \mathbb{R}$ *is absolutely continuous on* $[a,b]$.

(i) *If* $p \in [1,\infty)$ *then we have the inequality*

$$\|f\|_p^{\triangle} \le \begin{cases} \frac{2^{\frac{1}{p}}(b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}} \|f'\|_\infty, & if \quad f' \in L_\infty[a,b]; \\ \frac{(2\beta^2)^{\frac{1}{p}}(b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}} \|f'\|_\alpha, & if \quad f' \in L_\alpha[a,b], \\ & \qquad \alpha > 1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (b-a)^{\frac{2}{p}} \|f'\|_1, & if \quad f' \in L_1[a,b]. \end{cases} \tag{3.5}$$

(ii) *If* $p = \infty$, *then we have the inequality*

$$\|f\|_\infty^{\triangle} \le \begin{cases} (b-a)\|f'\|_\infty, & if\ f' \in L_\infty[a,b]; \\ (b-a)^{\frac{1}{\beta}}\|f'\|_\alpha, & if\ f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1. \end{cases} \tag{3.6}$$

PROOF. As $f : [a,b] \to \mathbb{R}$ is absolutely continuous , then $f(t) - f(s) = \int_s^t f'(u)du$ for all $t,s \in [a,b]$, and then

$$\begin{aligned} |f(t)-f(s)| &\\ = \left|\int_s^t f'(u)du\right| &\le \begin{cases} |t-s|\,\|f'\|_\infty & if \quad f' \in L_\infty[a,b]; \\ |t-s|^{\frac{1}{\beta}}\|f'\|_\alpha, & if \quad f' \in L_\alpha[a,b], \\ & \quad \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1 & if \quad f' \in L_1[a,b]. \end{cases} \end{aligned} \tag{3.7}$$

and so for $p \in [1,\infty)$, we may write

$$\begin{aligned} &|f(t)-f(s)|^p \\ &\le \begin{cases} |t-s|^p\,\|f'\|_\infty^p & \text{if } f' \in L_\infty[a,b]; \\ |t-s|^{\frac{p}{\beta}}\|f'\|_\alpha^p, & \text{if } f' \in L_\alpha[a,b],\ \alpha>1,\ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ \|f'\|_1^p & \text{if } f' \in L_1[a,b]. \end{cases} \end{aligned}$$

and then from (3.3), (3.4)

$$(3.8)\quad \|f\|_\infty^\Delta \le \begin{cases} \|f'\|_\infty \left(\int_a^b \int_a^b |t-s|^p dtds\right)^{\frac{1}{p}} & \text{if } f' \in L_\infty[a,b] \\ \|f'\|_\alpha \left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} dtds\right)^{\frac{1}{p}} & \text{if } f' \in L_\alpha[a,b], \\ & \quad \alpha > 1,\ \frac{1}{\alpha} + \frac{1}{\beta} = 1; \\ \|f'\|_1 \left(\int_a^b \int_a^b dtds\right)^{\frac{1}{p}} & \text{if } f' \in L_1[a,b]. \end{cases}$$

Further, since

$$\begin{aligned} \left(\int_a^b \int_a^b |t-s|^p dtds\right)^{\frac{1}{p}} &= \left[\int_a^b \left(\int_a^t (t-s)^p ds \int_t^b (s-t)^p ds\right) dt\right]^{\frac{1}{p}} \\ &= \left(\int_a^b \left[\frac{(t-a)^{p+1} + (b-t)^{p+1}}{p+1}\right] dt\right)^{\frac{1}{p}} \\ &= \frac{2^{\frac{1}{p}} (b-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}, \end{aligned}$$

giving

$$\left(\int_a^b \int_a^b |t-s|^{\frac{p}{\beta}} dtds\right)^{\frac{1}{p}} = \frac{(2\beta^2)^{\frac{1}{p}} (b-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}},$$

and

$$\left(\int_a^b \int_a^b dtds\right)^{\frac{1}{p}} = (b-c)^{\frac{2}{p}},$$

we obtain, from (3.8), the stated result (3.5).

Using (3.7) we have (for $p = \infty$) that

$$\|f\|_\infty^\Delta \le \begin{cases} \|f'\|_\infty \, ess \sup\limits_{(t,s)\in[a,b]^2} |t-s| \\ \|f'\|_\alpha \, ess \sup\limits_{(t,s)\in[a,b]} |t-s|^{\frac{1}{\beta}} \\ \|f'\|_1 \end{cases} = \begin{cases} (b-a)\|f'\|_\infty \\ (b-a)^{\frac{1}{\beta}} \|f'\|_\alpha \\ \|f'\|_1 \end{cases}$$

and the inequality (3.6) is also proved.

4. New Bounds for a Perturbed Generalised Taylor's Formula

Using the following pre-Grüss inequality (see[1, Lemma 1])

LEMMA 1 *Let $a \leq x$ and let $g, h : [a, x] \to \mathbb{R}$ be two integrable functions. If*

$$\phi \leq g(t) \leq \Phi \quad \text{for a.e. } t \in [a, x], \tag{4.1}$$

then

$$|T(g,h)| \leq \frac{1}{2}(\Phi - \phi)\sqrt{T(h,h)}, \tag{4.2}$$

where $T(\cdot,\cdot)$ is the Chebydhev functional on $[a, x]$, that is we recall

$$T(g,h) := \frac{1}{x-a}\int_a^x g(t)h(t)dt - \frac{1}{(x-a)^2}\int_a^x g(t)dt \int_a^x h(t)dt. \tag{4.3}$$

Matić, Pečarić and Ujević proved the following generalised Taylor's formula (see [1, Theorem 3]).

THEOREM 5 *Let $\{P_n\}_{n\in\mathbb{N}}$ be a harmonic sequence of polynomials. Let $I \subset \mathbb{R}$ be closed interval and $a \in I$. Suppose that $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalised Taylor's perturbed formula*

$$f(x) = \tilde{T}_n(f; a, x) + (-1)^n[P_{n+1}(x) - P_{n+1}(a)][f^{(n)}; a, x] + \tilde{G}_n(f; a, x), \tag{4.4}$$

where

$$\tilde{T}_n(f; a, x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)] \tag{4.5}$$

and

$$[f^{(n)}; a, x] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x - a}.$$

For $x \geq a$, the remainder $\tilde{G}(f; a, x)$ satisfies the estimate

$$|\tilde{G}_n(f; a, x)| \leq \frac{x-a}{2} \sqrt{T(P_n, P_n)} [\Gamma(x) - \gamma(x)], \tag{4.6}$$

where

$$\Gamma(x) : \sup_{t \in [a,x]} f^{(n+1)}(t) < \infty, \; \gamma(x) := \inf_{t \in [a,x]} f^{(n+1)}(s) > -\infty. \tag{4.7}$$

In the recent paper [2], by the use of the following representation

$$\begin{aligned} &\tilde{G}_n(f; a, x) \\ &= \frac{(-1)^n}{2(x-a)} \int_a^x \int_a^x (P_n(t) - P_n(s)) \Big(f^{(n+1)}(t) - f^{(n+1)}(s) \Big) dt ds, \end{aligned} \tag{4.8}$$

S.S. Dragomir improved the inequality (4.6) as follows

$$|\tilde{G}_n(f; a, x)| \leq (x-a) \sqrt{T(P_n, P_n)} \left[\frac{1}{x-a} \left| \left| f^{(n+1)} \right| \right|_2^2 - \left([f^{(n)}, a, x] \right)^2 \right]^{\frac{1}{2}}, \tag{4.9}$$

provided that $f^{(n)} \in L_2[a, x]$.

If $f^{(n)} \in L_\infty[a, x] \subset L_2[a, x]$(and the inclusion is strict), then indeed

$$\frac{1}{x-a} \left| \left| f^{(n+1)} \right| \right|_2^2 - \left([f^{(n)}, a, x] \right)^2 \leq \frac{1}{4} (\Gamma(x) - \gamma(x))^2$$

and so (4.9) is a refinement of (4.6).

In this paper, we point out some other bounds for the remainder $G(f; a, x)$ in terms of the seminorms $||P_n||_q^{\Delta}, \left| \left| f^{(n+1)} \right| \right|_p^{\Delta}$ where $p, q \in [1, \infty]$.

The following theorem holds.

THEOREM 6 *Let $f : I \to \mathbb{R}$ be such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the representation (4.4) and for $x \geq a$ we have an estimate for the remainder given by:*

(4.10)

$$\|\tilde{G}_n(f;a,x)\| \leq \begin{cases} \dfrac{1}{2(x-a)}\|P_n\|_1^\Delta \|f^{(n+1)}\|_\infty^\Delta & \text{if} \quad f^{(n+1)} \in L_\infty[a,x]; \\ \dfrac{1}{2(x-a)}\|P_n\|_q^\Delta \|f^{(n+1)}\|_p^\Delta & \text{if} \quad f^{(n+1)} \in L_p[a,x]; \\ \dfrac{1}{2(x-a)}\|P_n\|_\infty^\Delta \|f^{(n+1)}\|_1^\Delta & \text{if} \quad f^{(n+1)} \in L_1[a,x]. \end{cases}$$

Moreover,

$$\|f^{(n+1)}\|_p^\Delta \leq \begin{cases} \frac{2^{\frac{1}{p}}(x-a)^{1+\frac{2}{p}}}{[(p+1)(p+2)]^{\frac{1}{p}}}\|f^{(n+2)}\|_\infty & \text{if} \quad f^{(n+2)} \in L_\infty\left[a,x\right]; \\ \frac{(2\beta^2)^{\frac{1}{p}}(x-a)^{\frac{1}{\beta}+\frac{2}{p}}}{[(p+\beta)(p+2\beta)]^{\frac{1}{p}}}\|f^{(n+2)}\|_\alpha & \text{if} \quad f^{(n+2)} \in L_\alpha\left[a,x\right], \\ & \qquad \alpha > 1, \ \frac{1}{\alpha}+\frac{1}{\beta}=1; \\ (x-a)^{\frac{2}{p}}\|f^{(n+2)}\|_1, & \text{if} \quad f^{(n+2)} \in L_1\left[a,x\right], \end{cases}$$

where p $\in [1,\infty)$ and

$$\|f^{(n+1)}\|_\infty^\Delta \leq \begin{cases} (x-a)\|f^{(n+2)}\|_\infty & \text{if} \quad f^{(n+2)} \in L_\infty\left[a,x\right]; \\ (x-a)^{\frac{1}{\beta}}\|f^{(n+2)}\|_\alpha & \text{if} \quad f^{(n+2)} \in L_\alpha\left[a,x\right], \\ & \qquad \alpha > 1, \ \dfrac{1}{\alpha}+\dfrac{1}{\beta}=1; \\ \|f^{(n+2)}\|_1 & \text{if} \quad f^{(n+2)} \in L_1\left[a,x\right], \end{cases}$$

(4.11)

$$\|P_n\|_q^\Delta \leq \begin{cases} \dfrac{2^{\frac{1}{q}}(x-a)^{1+\frac{2}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}\|P_{n-1}\|_\infty; & \\ \dfrac{(2\gamma^2)^{\frac{1}{q}}(x-a)^{\frac{1}{\gamma}+\frac{2}{q}}}{[(q+\gamma)(q+2\gamma)]^{\frac{1}{q}}}\|P_{n-1}\|_\delta & \text{if} \quad \gamma,\delta > 1, \ \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1; \\ (x-a)^{\frac{2}{q}}\|P_{n-1}\|_1 & \text{if} \quad f^{(n+2)} \in L_1\left[a,x\right], \end{cases}$$

where $q \in [1, \infty)$ and

$$\|P_n\|_{\infty}^{\Delta} \leq \begin{cases} (x-a)\|P_{n-1}\|_{\infty}; \\ (x-a)^{\frac{1}{\gamma}}\|P_{n-1}\|_{\delta} & \text{if} \quad \gamma, \delta > 1, \ \frac{1}{\gamma} + \frac{1}{\delta} = 1; \\ \|P_{n-1}\|_1. \end{cases} \tag{4.12}$$

PROOF Use the representation (4.8) to get

$$\left|\tilde{G}_n(f; a, x)\right| \leq \frac{1}{2(x-a)} \int_a^x \int_a^x |P_n(t) - P_n(s)| \left|f^{(n+1)}(t) - f^{(n+1)}(s)\right| dtds.$$

It is obvious that if $f^{(n+1)} \in L_{\infty}[a, x]$, then

$$\begin{aligned} M :&= \int_a^x \int_a^x |P_n(t) - P_n(s)| \left|f^{(n+1)}(t) - f^{(n+1)}(s)\right| dtds \\ &\leq ess \sup_{(t,s)\in[a,b]^2} \left|f^{(n+1)}(t) - f^{(n+1)}(s)\right| \int_a^x \int_a^x |P_n(t) - P_n(s)| \, dtds \\ &= \|P_{n1}^{\Delta}\| \|f^{(n+1)}\|_{\infty}^{\Delta}. \end{aligned}$$

Using Hölder's integral inequality for double integrals, we have

$$\begin{aligned} M &\leq \left(\int_a^x \int_a^x |P_n(t) - P_n(s)|^q \, dtds\right)^{\frac{1}{q}} \left(\int_a^x \int_a^x \left|f^{(n+1)}(t) - f^{(n+1)}(s)\right|^p dtds\right)^{\frac{1}{p}} \\ &= \|P_{nq}^{\Delta}\| \|f^{(n+1)}\|_p^{\Delta} \end{aligned}$$

and, similarly,

$$M \leq \|P_{n\infty}^{\Delta}\| \|f^{(n+1)}\|_1^{\Delta}.$$

For the second part of the theorem, we apply Theorem 4 for $f^{(n+1)}$ and P_n, taking into account the fact that $P_n'(t) = P_{n-1}(t)$.

COROLLARY 5 *Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Moreover, suppose that $f : I \to \mathbb{R}$ is such that $f^{(n)}$ is absolutely continuous. Then for $x \in I$ and $\lambda \in [0,1]$ we have*

$$
\begin{aligned}
(4.13)\quad \Bigg| f(x) - f(a) - \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \\
\times \left[\lambda^k f^{(k)}(x) + (-1)^{k+1} (1-\lambda)^k f^{(k)}(a) \right] \\
+ (-1)^{n+1} \frac{(x-a)^n}{(n+1)!} \left[\lambda^{n+1} + (-1)^n (1-\lambda)^{n+1} \right] \left[f^{(n)}(x) - f^{(n)}(a) \right] \Bigg|
\end{aligned}
$$

$$
\leq \begin{cases}
B_1 \| f^{(n+1)} \|_{\infty}^{\Delta} & \text{if} \quad f^{(n+1)} \in L_\infty [a,x]\,; \\
B_q \| f^{(n+1)} \|_{p}^{\Delta} & \text{if} \quad f^{(n+1)} \in L_p [a,x]\,, \\
 & \qquad p > 1, \ \frac{1}{p} + \frac{1}{q} = 1; \\
B_\infty \| f^{(n+1)} \|_{1}^{\Delta} & \text{if} \quad f^{(n+1)} \in L_1 [a,x]\,,
\end{cases}
$$

where, *for* $1 \leq q < \infty$

$$
(4.14) \quad B_q = \begin{cases}
b_{1,q} \frac{(x-a)^{n-1}}{(n-1)!} \left[\frac{1}{2} + \left| \lambda - \frac{1}{2} \right| \right]^{n-1}, \\
b_{\gamma,q} \dfrac{(x-a)^{n-1-\frac{1}{\delta}}}{(n-1)!} \left[\dfrac{(1-\lambda)^{(n-1)\delta+1} + \lambda^{(n-1)\delta+1}}{(n-1)\delta+1} \right]^{\frac{1}{\delta}}, \\
\qquad \dfrac{1}{\gamma} + \dfrac{1}{\delta} = 1, \quad \gamma, \delta > 1, \\
b_{\infty,q} \frac{(x-a)^n}{n!} \left[(1-\lambda)^n + \lambda^n \right],
\end{cases}
$$

with

$$
(4.15) \quad \begin{cases}
b_{\gamma,q} = \dfrac{2^{\frac{1}{q}-1} \gamma^{\frac{2}{q}} (x-a)^{\frac{1}{\gamma}+\frac{2}{q}-1}}{\left[(q+\gamma)(q+2\gamma)\right]^{\frac{1}{q}}}, \quad 1 \leq \gamma < \infty \quad \text{and} \\
b_{\infty,q} = \frac{1}{2} (x-a)^{\frac{2}{q}-1},
\end{cases}
$$

and

$$(4.16)\quad B_\infty = \begin{cases} \frac{1}{2}\frac{(x-a)^{n-1}}{(n-1)!}\left[\frac{1}{2}+\left|\lambda-\frac{1}{2}\right|\right]^{n-1}, \\ \frac{1}{2}\frac{(x-a)^{n-1}}{(n-1)!}\left[\frac{(1-\lambda)^{(n-1)\delta+1}+\lambda^{(n-1)\delta+1}}{(n-1)\delta+1}\right]^{\frac{1}{\delta}}, \frac{1}{\gamma}+\frac{1}{\delta}=1, \gamma,\delta>1 \\ \frac{1}{2}\frac{(x-a)^n}{n!}\left[(1-\lambda)^n+\lambda^n\right]. \end{cases}$$

PROOF Rearranging identity (4.4) and taking the modulus gives the remainder for the perturbed generalised Taylor series $\left|\tilde{G}(f;a,x)\right|$ bounded by expression (4.10). Taking $P_n(t)$ as given by (P) readily produces the left hand side of (4.13).

Now for the bound. From (4.10) with $P_n(t)$ as given by (P), we have $\|P_n\|_q^\Delta$ and $\|P_n\|_\infty^\Delta$ defined in terms of $\|P_{n-1}\|_\delta$ and $\|P_{n-1}\|_\infty$ in equations (4.11) and (4.12) which are as defined in (2.4) and (2.5) respectively. Thus, substitution of (2.4) and (2.5) into (4.11) gives, on utilizing (4.10), the first two inequalities in (4.13) with B_q being defined by (4.14) for $1 \le q < \infty$. Substitution of (2.4) and (2.5) into (4.12) gives, on substitution into (4.10), the last inequality in (4.13) where B_∞ is as given by (4.16).

COROLLARY 6 *Let the assumptions of Corollary 5 hold. Then the following inequality is valid. Namely, for $f^{(n+1)}$ in the obvious Lebesgue norm on $[a,x]$*

$$(4.17)\quad \left| f(x) - f(a) - \sum_{k=1}^{n}(-1)^{k+1}\frac{(x-a)^k}{2^k k!}[f^{(k)}(x)+(-1)^{k+1}f^{(k)}(a)] \right.$$
$$\left. + \frac{[(-1)^{n+1}+1]}{2^{n+1}(n+1)!}(x-a)^n[f^{(n)}(x)-f^{(n)}(a)] \right|$$

$$\le \begin{cases} \tilde{B}_1\|f^{(n+1)}\|_\infty^\Delta; \\ \tilde{B}_q\|f^{(n+1)}\|_p^\Delta; \\ \tilde{B}_\infty\|f^{(n+1)}\|_1^\Delta, \end{cases}$$

where, for $1 \le q < \infty$,

$$\tilde{B}_q = \begin{cases} b_{1,q} \dfrac{(x-a)^{n-1}}{2^{n-1}(n-1)!} \\ b_{\gamma,q} \dfrac{(x-a)^{n-1+\frac{1}{\gamma}}}{2^{n-1}\left((n-1)\delta+1\right)^{\frac{1}{\delta}}(n-1)!}, \quad \dfrac{1}{\gamma}+\dfrac{1}{\delta}=1, \ \gamma,\delta>1 \\ b_{\infty,q} \dfrac{(x-a)^{n}}{2^{n-1}n!} \\ b_{\ ,q} \text{ are as defined by (4.15)} \end{cases}$$

and

$$\tilde{B}_\infty = \begin{cases} \frac{(x-a)^{n-1}}{2^n(n-1)!}; \\ \frac{(x-a)^{n-1}}{2^n[(n-1)\delta+1]^{\frac{1}{\delta}}(n-1)!}, \quad \frac{1}{\gamma}+\frac{1}{\delta}=1, \ \gamma,\delta>1; \\ \frac{(x-a)^{n-1}}{2^n n!}. \end{cases}$$

PROOF. Trivial. Taking $\lambda = \frac{1}{2}$ in Corollary 5 readily gives the desired results.

REMARK 2 *It should be re-emphasised here as stated after Theorem 2 that the sharpest bound that can be obtained by restricting λ in (4.13) is to take $\lambda = \frac{1}{2}$. That is, the results of Corollary 6. Furthermore, for n even it may be noticed from (4.17) that there is no perturbation in the Taylor series.*

In addition, note that coarser bounds may be obtained by using the results of Theorem 4 or 6 and so $\left\|f^{(n+1)}\right\|_p^{\Delta} \le K_{p,\beta}\left\|f^{(n+2)}\right\|_\alpha$, where $p \ge 1$ and $\alpha \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $\frac{1}{\alpha}+\frac{1}{\beta}=1$.

COROLLARY 7 *Let $I \subset \mathbb{R}$ be a closed interval and $a \in I$. Further, suppose that $f : I \to \mathbb{R}$ and is such that $f^{(n)}$ is absolutely continuous. Then we have for $f^{(n+1)}$ in the obvious Lebesgue norm on $[a,x]$*

$$(4.18)\quad \left| f(x) - f(a) - \sum_{k=1}^{n+1} \frac{(x-a)^k}{k!} f^{(k)}(a) + (-1)^{n+1} \frac{(x-a)^n}{(n+1)!} f^{(n+1)}(x) \right|$$

$$\leq \begin{cases} B_1^* \left\| f^{(n+1)} \right\|_\infty^\Delta ; \\ B_q^* \left\| f^{(n+1)} \right\|_p^\Delta ; \\ B_\infty^* \left\| f^{(n+1)} \right\|_1^\Delta , \end{cases}$$

where, for $1 \leq q < \infty$

$$B_q^* = \begin{cases} b_{1,q} \frac{(x-a)^{n-1}}{(n-1)!} ; \\ \\ b_{\gamma,q} \frac{(x-a)^{n-1+\frac{1}{\delta}}}{(n-1)!((n-1)\delta+1)^{\frac{1}{\delta}}}, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1, \quad \gamma, \delta > 1; \\ \\ b_{\infty,q} \frac{(x-a)^n}{n!} \end{cases}$$

with $b_{\gamma,q}$ *and* $b_{\infty,q}$ *as given by (4.15). Further,*

$$B_\infty^* = \begin{cases} \frac{(x-a)^{n-1}}{2(n-1)!} ; \\ \\ \frac{(x-a)^{n-1}}{2(n-1)![(n-1)\delta+1]^{\frac{1}{\delta}}}, \quad \frac{1}{\gamma} + \frac{1}{\delta} = 1, \quad \gamma, \delta > 1; \\ \\ \frac{(x-a)^{n-1}}{2\,n!} . \end{cases}$$

PROOF Taking $\lambda = 0$ in Corollary 5 produces the results as stated after some minor manipulation.

REMARK 3. *The result (4.18) represents a perturbed Taylor series expansion about a point* $x = a$ *together with the evaluation of* $f^{(n+1)}(x)$ *involved in the perturbation. The bounds are given in terms of the Lebesgue p-norms on* $[a,b]^2$ *for* $f_\Delta : [a,b]^2 \to \mathbb{R}$ *where* $f_\Delta(t,s) = f(t) - f(s)$, *as introduced in the current work.*

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