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STOCHASTIC INEQUALITIES IN TWO REPAIRABLE UNITS

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Abstract. In this paper we investigated a replacement model with two types of repairs. Repairs are classified into minimal and perfect repair. An operating unit is completely replaced whenever it reaches age $\tau(\tau > 0)$(planned replacement). If it fails at age $t < \tau$, it is either restored by a entire unit with probability $p(t)$(perfect repair), or it undergoes minimal repair with probability $\overline{p}(t) = 1 - p(t)$. After a planned replacement, the procedure is repeated.

1. Introduction and Basic Inequalities

Let X and X' be two non-negative independent random variables with survival functions $\overline{F}$ and $\overline{G}$ respectively, the statement that random variable X is stochastically greater than the random variable X', that is, the first unit with failure time X is said to be more reliable than the second unit with failure time Y', if

$$\overline{F}(t) = p[X > t] \geq p[X' > t] = \overline{G}(t) \quad \text{for all } \ t \in \mathbb{R}, \tag{1.1}$$

and the relation is written $X \overset{st}{\geq} X'$. The physical interpretation of the inequality (1.1) is that the first unit is better than the second unit if the first unit is more reliable than the second unit.

An extension to n-component random vectors is obtained by calling random vector $\mathbf{X} = (X_1, \ldots, X_n)$ stochastically greater than random vector $\mathbf{X}' = (X_1', \ldots, X_n')$, written $\mathbf{X} \overset{st}{\geq} \mathbf{X}'$, if $\Phi\mathbf{X} \geq \Phi\mathbf{X}'$ for all increasing functions Φ. (A Borel-measurable function

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$\Phi : \mathbb{R}^n \to \mathbb{R}$ is called increasing if $\mathbf{x} \geq \mathbf{x}'$, for any two vectors $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{x}' = (x'_1, \ldots, x'_n)$, written $\mathbf{x} \geq \mathbf{x}'$ if $x_i \geq x'_i$, $i = 1, \ldots, n$, implies that $\Phi(x_1, \ldots, x_n) \geq \Phi(x'_1, \ldots, x'_n)$.)

As a further extension of stochastic ordering to stochastic processes, we call stochastic process $\{X(t),\ t \geq 0\}$ stochastically greater than stochastic process $\{X'(t),\ t \geq 0\}$, written $\{X(t),\ t \geq 0\} \overset{st}{\geq} \{X'(t), t \geq 0\}$ if

$$\{X(t_1), \ldots, X(t_n)\} \overset{st}{\geq} \{X'(t_1), \ldots, X'(t_n)\}, \tag{1.2}$$

for all $0 \leq t_1 <, \ldots, < t_n$ and all $n = 1, 2, \ldots$, implies that $\Phi(\{X(t),\ t \geq 0\}) \overset{st}{\geq} \Phi(\{X'(t),\ t \geq 0\})$ for if $x_t \geq x'_t$ implies $\Phi x(\cdot) \geq \Phi x'(\cdot)$.

LEMMA 1.1. *Let* $\mathbf{X}$ *is independent and* $\mathbf{X}'$ *is independent.* $\mathbf{X} \overset{st}{\geq} \mathbf{X}'$ *if there exist two random vectors* $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$ *and* $\hat{\mathbf{X}}' = (\hat{X}'_1, \ldots, \hat{X}'_n)$ *defined a common probability space such that* $p[\hat{\mathbf{X}} \geq \hat{\mathbf{X}}'] = 1$ *and* $\mathbf{X} \overset{st}{=} \hat{\mathbf{X}}$ *and* $\mathbf{X}' \overset{st}{=} \hat{\mathbf{X}}'$, *that is,* $\hat{\mathbf{X}}$ *have the same distribution of* $\mathbf{X}$ *and* $\hat{\mathbf{X}}'$ *have the same distribution of* $\mathbf{X}'$.

Proof. This result has been proved by Ross [17] pp. 255-256.

LEMMA 1.2. *Let* $\mathbf{X}$ *is independent and* $\mathbf{X}'$ *is independent.* $\mathbf{X}$ *stochastically greater than* $\mathbf{X}'$ *written* $\mathbf{X} \overset{st}{\geq} \mathbf{X}'$ *if* $\Phi\mathbf{X} \geq \Phi\mathbf{X}'$ *where* Φ *is increasing functions.*

Proof. From Lemma 1.1, let $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$ is independent and $\mathbf{Y}$ is independent, where $\hat{X}_i$ has the same distribution of X_i and $\hat{X}_i \leq X'_i$. Then $\Phi\hat{\mathbf{X}} \geq \Phi\mathbf{X}'$. For each

$$\Phi(\hat{X}_1, \ldots, \hat{X}_n) \supseteq \Phi(Y_1, \ldots, Y_n),$$

and so

$$p[\Phi\hat{\mathbf{X}} > t] \geq p[\Phi\mathbf{Y} > t]. \tag{1.3}$$

Note that the left-hand side of (1.3) is equal to $p[\Phi\mathbf{X} > t]$.

LEMMA 1.3. *Let* $\mathbf{X}$ *is independent and* $\mathbf{Y}$ *is independent.* $\mathbf{X}$ *stochastically greater than* $\mathbf{Y}$ *written* $\mathbf{X} \overset{st}{\geq} \mathbf{Y}$ *if* $E[\Phi\mathbf{X}] \geq E[\Phi\mathbf{Y}]$ *for all increasing functions* Φ *for which the expectations exist.*

Proof. Suppose that $X_i \overset{st}{\geq} Y_i$, $i = 1, 2$, with X_1 and X_2 are independent, and Y_1 and Y_2 are independent. Then

$$(X_1, X_2) \geq (Y_1, Y_2).$$

Let $\Phi(x_1, x_2)$ is an increasing function such that $E[(X_1, X_2)]$ and $E[(Y_1, Y_2)]$ exist. From Lemma 1.1, let $\hat{X}_1$ has the same distribution of X_1. Then

$$\begin{aligned} E[\Phi(X_1,\ X_2|X_1 = x_1)] &= E[\Phi(\hat{X}_1, X_2|\hat{X}_1 = x_1)] \\ &\geq E[\Phi(\hat{X}_1, Y_2|\hat{X}_1 = x_1)]. \end{aligned}$$

It follows that

$$E[\Phi(X_1, X_2)] \geq E[\Phi(\hat{X}_1, Y_2)]. \tag{1.4}$$

Inequalities (1.4) can be also expressed by

$$E[\Phi(\hat{X}_1, Y_2)] \geq E[\Phi(Y_1, Y_2)]. \tag{1.5}$$

Combing the (1.4)-(1.5) inequalities, we obtain that

$$E[\Phi(X_1, X_2)] \geq E[\Phi(Y_1, Y_2)]. \tag{1.6}$$

In case Φ is increasing, it is clear from the above proof that equality holds if and only if $\mathbf{X} = \mathbf{Y}$.

2. Inequalities in Series unit

In this section, we considered the basic inequalities involving expectations of increasing functions of an n-unit series unit before proceeding to applications of repairable unit in the following section. And we now show that the mean life of an n-unit series unit with IFR units whose means are $1/\mu_i$ $(i = 1, \ldots, n)$ exceds the mean life of an n-unit series unit with exponential and means $1/\mu_i$ $(i = 1, \ldots, n)$.

PROPOSITION 2.1. *Let $X_i(Y_i)$ the life length of unit i, have continuous distribution $F_i(G_i)$ with mean μ_i. Let $\mathbf{X}(\mathbf{Y})$ is independent and $F_i \underset{*}{<} G_i$, $i = 1, \ldots, n$, i.e., F_i is star-shaped with respect to $G_i(F_i \underset{*}{<} G_i$ if $(1/x)(G^{-1}F_i(x))$ is increasing for $x \geq 0)$. Then the mean life of a series unit using units with lives $\mathbf{X}$ is stochastically greater than the corresponding unit mean life using units with lives $\mathbf{Y}$, written $\mathbf{X} \overset{st}{\geq} \mathbf{Y}$ that is,*

$$E[\min \mathbf{X}] \geq E[\min \mathbf{Y}]. \tag{2.1}$$

Proof. See Barlow and Proschan [3] pp. 122.

LEMMA 2.1. *Let F is IFRA with mean $1/\mu$ and $\overline{G}(x) = \exp(-\mu x)$ such that $F \underset{*}{<} G$ and $E[X] = E[Y]$. Then*

$$\int_0^t \overline{F}(x)\, dx \geq \int_0^t \overline{G}(x)\, dx, \tag{2.2}$$

and

$$\int_t^{+\infty} \overline{F}(x)\, dx \leq \int_t^{+\infty} \overline{G}(x)\, dx. \tag{2.3}$$

Proof. Assume that F is IFRA and F is not identically equal to G. Because F have IFRA distribution and G have exponential distribution with the same mean, $\overline{F}$ crosses $\exp(-x/\mu)$ exactly once, and the crossing in necessarily from above at, say x_0; that is $\overline{F}(x_0) = \overline{G}(x_0)$.

PROPOSITION 2.2. *(a) Let F is IFRA distribution with mean $1/\mu$ and $\overline{G}(x) = \exp(-\mu x)$ such that $F \underset{*}{<} G$ and $E[X] = E[Y]$. Then,*

$$E[\Phi(X)] \leq E[\Phi(Y)], \tag{2.4}$$

where $\Phi(X)$ denote the decreasing convex function.

(b) Let F is IFRA distribution with mean $1/\mu$ and $\overline{G}(x) = \exp(-\mu x)$ such that $F \underset{*}{<} G$ and $E[X] = E[Y]$. Let $\Phi(X)$ is decreasing convex functions on $[0, +\infty)$. Then

$$E[\Phi \min \mathbf{X}] \leq E[\Phi \min \mathbf{Y}]. \tag{2.5}$$

Proof. (a) Let $X = \min \mathbf{X}$.

$$E[\Phi(X)] = \int_0^{+\infty} \Phi(x) dF(x)\, dx$$
$$= -\phi(x)\overline{F}(x)\Big|_0^{+\infty} + \int_0^{+\infty} \phi(x)\overline{F}(x)\, dx,$$

where we denoted $\phi(x) = \Phi'(x)$ and $\overline{F}_i(x) = 1 - F_i(x)$. Since $\Phi(X)$ is bounded then $-\Phi(x)\overline{F}(x)\Big|_0^{+\infty} < +\infty$. Moreover, since $\Phi(X)$ is convex then $\phi(x)$ is increasing and thus all the conditionals of Theorem 4.8[2] are satisfied and

$$\int_0^{+\infty} \phi(x)\overline{F}(x)\, dx \leq \int_0^{+\infty} \phi(x)\overline{G}(x)\, dx. \tag{2.6}$$

Since $\overline{F}$ crosses $\overline{G}$ exactly once, and the crossing in necessarily from above at, say x_0; that is $\overline{F}(x_0) = \overline{G}(x_0)$. Then

$$\int_0^{+\infty} \phi(x)\overline{F}(x)\, dx - \int_0^{+\infty} \phi(x)\overline{G}(x)\, dx$$
$$= \int_0^{x_0} [\phi(x) - \phi(x_0)][\overline{F}(x) - \overline{G}(x)]\, dx$$
$$+ \int_{x_0}^{+\infty} [\phi(x) - \phi(x_0)][\overline{F}(x) - \overline{G}(x)]\, dx \leq 0.$$

is nonpositive, since for $\phi(x) - \phi(x_0) \geq 0$ and $\overline{F}(x) - \overline{G}(x) \leq 0$ for $x_0 \leq x < +\infty$. Similarly, $\phi(x) - \phi(x_0) \leq 0$ and $\overline{F}(x) - \overline{G}(x) \geq 0$ for $x_0 \leq x < +\infty$.

(b) The proof is similar to part (a).

PROPOSITION 2.3. *If the X_i are independent, F_i is IFR distribution with mean $1/\mu_i$, and $G_i(x) = 1 - \exp(-\mu_i x)$ for $i = 1, \ldots, n$ such that*

$$E[\min \mathbf{X}] \geq \frac{1}{\sum_{i=1}^{n} \frac{1}{E[X_i]}}. \tag{2.7}$$

Proof. For any nonnegative random variables X_i, $i = 1, \dots, n$, $\min \mathbf{X} = 1/(\max(1/\mathbf{X}))$, and

$$E[\min \mathbf{X}] = E\left[\frac{1}{\max\left(\frac{1}{\mathbf{X}}\right)}\right] \geq \frac{1}{E\left[\max\left(\frac{1}{\mathbf{X}}\right)\right]} \geq \frac{1}{\sum_{i=1}^{n} E\left[\frac{1}{X_i}\right]} \geq \frac{1}{\sum_{i=1}^{n} \frac{1}{E[X_i]}}.$$

The second and forth inequality are just an application of Jensen's inequality. The third inequality is $\max \mathbf{X} \leq \sum_{i=1}^{n} X_i$.

PROPOSITION 2.4. *Let $X = \min \mathbf{X}$. If X_i are independent, F_i is IFR distribution, and G_i are exponential, for $i = 1, \dots, n$, $E[X_i] = E[Y_i] = 1/\mu_i$. Then for $\Phi(x) = \exp(-\alpha x)$,*

$$E[\phi(X)] = E[\exp(-\alpha X)] = E[\exp(-\alpha \min \mathbf{X})] \leq \frac{\sum_{i=1}^{n} \mu_i}{\alpha + \sum_{i=1}^{n} \mu_i}. \tag{2.8}$$

Proof. The distribution of X is given by $\overline{F}_X(x) = \prod_{i=1}^{n} \overline{F}_i(x)$ and its density by $f_X(x)$. Therefore,

$$E[\exp(-\alpha X)] \leq 1 - \alpha \int_0^{+\infty} \prod_{i=1}^{n+1} \overline{G}_i(x)\, dx, \tag{2.9}$$

where the inequality (2.9) follows from Proposition 2.3 and $\overline{G}_i(x) = \exp(-\mu_i x)$ for $i = 1, \dots, n$ and $\overline{G}_{n+1}(x) = \exp(-\alpha x)$

$$E[\exp(-\alpha X)] \leq 1 - \frac{\alpha}{\alpha + \sum_{i=1}^{n} \mu_i}. \tag{2.10}$$

3. Some models

Let t be the age of a unit and $\lambda(t)$ be the failure rate(or the hazard rate) function belonging to $F(t)$; that is,

$$\lambda(t) = \frac{f(t)}{1 - F(t)}, \quad f(t) = \frac{d}{dt} F(t). \tag{3.1}$$

In general, the failure rate of a unit after repair is not necessarily the same as just before the failure, but it tends to vary a little depending on the numbers and the degrees of failures. We assumed that the good-as-new condition of unit is described by a survival distribution $\overline{F}$ which is a continuous function and such that $\overline{F}(x) > 0$ for all $t > 0$.

Let $G(t)$ be the distribution function of random variable $Y =$ total time among perfect repairs. Formally, $Y_1, Y_2, \dots, Y_{n-1}$ be a sequence of nonnegative independent random variables with a common distribution $G(t)$. Also let $G(t) = p[Y \leq t]$. The survival distribution of the time between successive perfect repairs is given by

$$\overline{G}(t) = \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right], \tag{3.2}$$

where $\overline{G}(t) = 1 - G(t)$ and $G(0) = 0$. Hence, $p(t)\lambda(t)$ denotes the failure rate function belonging to $G(t)$. Thus if $F(t)$ is the failure time distribution following perfect repair, then the failure time distribution following minimal repair for a unit with fails at age x given by

$$\overline{F}_t(x) = p[X_t > x] = p[X > t + x | X > t] = \frac{\overline{F}(t + x)}{\overline{F}(t)}. \tag{3.3}$$

Model 3.1. An operating unit is repaired at failure. Repairs are classified into minimal and perfect repair. A minimal repair is the maintenance active to repair the failed unit so that its function is recovered without changing its age, while a perfect repair restores the entire unit into the new condition so that it behaves as a new unit. After a failure occurs, repair are perfect with probability $p(t)$, and minimal with probability $\overline{p}(t) = 1 - p(t)$. If we let $N(t)$ denote the number of perfect repairs in $[0,\ t]$, then the process $N = \{N(t),\ t \geq 0\}$ is a renewal process generated by the life distribution G of a new unit. We denote the number of minimal repairs up to tome t by $N_m(t)$. It is well known that the process $N_m = \{N_m(t), t \geq 0\}$ is a Nonhomogeneous Possion process(NHPP) with mean function $E[N_m(t)] = -\ln \overline{F}(t)$, where $\overline{F}(t) = 1 - F(t)$ is the survival probability and $F(t)$ is assumed to be continuous and strictly between 0 and 1 for all values of $t \geq 0$. Then the hazard function Λ is defined by

$$\Lambda(t) = -\log \overline{F}(t),$$

i.e., $\overline{F}(t) = \exp[-\Lambda(t)]$, for $t \geq 0$.

PROPOSITION 3.1. *(a) Let* $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{X}' = (X_1', \ldots, X_n')$ *be two sequences of non-negative random variables such that* $X_1 \overset{st}{\geq} X_1'$,

$$X_{i+1}|_{(X_j,\ j=1,\ldots,i)=(x_j,\ j=1,\ldots,i)} \overset{st}{\geq} X'_{i+1}|_{(X'_j,\ j=1,\ldots,i)=(x_j,\ j=1,\ldots,i)}$$

for $i = 1, \ldots, n-1$ *and for all* $x_2, \ldots, x_n$ *and*

$$X_{i+1}|_{(X_j,\ j=1,\ldots,i)=(x_j,\ j=1,\ldots,i), i=1,\ldots,n-1}$$

and

$$X'_{i+1}|_{(X'_j,\ j=1,\ldots,i)=(x_j,\ j=1,\ldots,i), i=1,\ldots,n-1}$$

are stochastically increasing in the x*'s. Then*

$$\mathbf{X} \overset{st}{\geq} \mathbf{X}'.$$

(b) Let $\mathbf{X} = (X_1, \ldots, X_n)$ *and* $\mathbf{X}' = (X_1', \ldots, X_n')$ *be two sequences of non-negative random variables with the life times of the two processes* $\{N_m(t),\ t \geq 0\}$ *and* $\{N'_m(t),\ t \geq 0\}$*. Let we assumed that*

$$\mathbf{X} \overset{st}{\geq} \mathbf{X}'.$$

Then for all $0 \le t_1 <, \dots, < t_n$ *and* $n = 1, 2, \dots,$

$$\{N_m(t_1), \dots, N_m(t_n)\} \le \{N'_m(t_1), \dots, N'_m(t_n)\}.$$

Proof. (a) This result has been proved by Theorem 4.13 of Barlow and Proschan[3].

(b) By Lemma 1.1, there is a common probability space and random variables $\hat{X}$ and $\hat{X}'$ defined on it such that $\hat{X} \ge \hat{X}'$, and $X \stackrel{st}{=} \hat{X}'$ and $X' \stackrel{st}{=} \hat{X}'$. Now let $\{\hat{N}_m(t),\ t \ge 0\}$ and $\{\hat{N}'_m(t),\ t \ge 0\}$ be the process with life times $\hat{X}$ and $\hat{X}'$. Then

$$\begin{aligned}\{\hat{N}'_m(t) = k\} \subseteq \{\hat{N}'_m(t) \ge k\} &= \left\{\sum_{i=1}^{k} \hat{X}'_i \le t\right\} \\ &\subseteq \left\{\sum_{i=1}^{k} \hat{X}_i \le t\right\} \\ &= \{\hat{N}_m(t) \ge k\} = \{\hat{N}_m(t) = k\}.\end{aligned}$$

Thus $\hat{N}'_m(t) \le \hat{N}_m(t),\ t \ge 0$. Finally

$$\begin{aligned}(N'_m(t_i),\ i = 1, \dots, n) &= (\hat{N}'_m(t_i),\ i = 1, \dots, n) \\ &\le (\hat{N}_m(t_i),\ i = 1, \dots, n) \\ &= (N_m(t_i),\ i = 1, \dots, n).\end{aligned}$$

PROPOSITION 3.2. *Let* $F(t)$ *be NBU. Let* $N(t)$ *be renewal processes generated by the life distributions* $G(t)$. *The process* $N_m(t)$ *is a NHPP with mean function* $E[N_m(t)] = -\ln \overline{F}(t)$ *where* $\overline{F}(t) = 1 - F(t)$ *is the survival function of the unit. Then*

$$N_m(t) \stackrel{st}{\ge} N(t). \tag{3.5}$$

Proof. Let $F(t)$ be NBU. Here we let $\{X_1, \dots, X_k\}$ and $\{Y_1, \dots, Y_n\}$ denote the life length for the minimal repair $N_m(t)$ and the renewal $N(t)$, respectively. Since $p[Y_1 > t] = \overline{G}(t) \geq p[X_1 > t] = \overline{F}(t)$, and

$$p[X_k > t | X_1 = x_1, \ldots, X_{k-1} = x_{k-1}] = \overline{F}_{\sum_{i=1}^{k-1} x_i}(t),$$

where

$$\overline{F}_{\sum_{i=1}^{k-1} x_i}(t) = \frac{\overline{F}\left(\sum_{i=1}^{k-1} x_i + t\right)}{\overline{F}\left(\sum_{i=1}^{k-1} x_i\right)}.$$

And

$$p[Y_n > t | Y_1 = y_1, \ldots, Y_{n-1} = y_{n-1}] = p[Y_n > t] = \overline{G}(t).$$

By the NBU

$$\frac{\overline{F}\left(\sum_{i=1}^{k-1} x_i + t\right)}{\overline{F}\left(\sum_{i=1}^{k-1} x_i\right)} \leq \overline{G}(t).$$

PROPOSITION 3.3[AGE-NONDEPENDENT MINIMAL REPAIR]. *Let $F(t)$ be NBUE. An operating unit is completely replaced whenever it reaches age $\tau (\tau > 0)$. If it fails at age $t < \tau$, it is either restored by a entire unit with probability $p(t)$, or it undergoes minimal repair with probability $\overline{p}(t) = 1 - p(t)$. Then*

$$E[W] \leq \frac{\int_0^\tau \overline{F}(t)\,dt \left[\overline{G}(\tau) + \int_0^\tau \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right] \lambda(t)\,dt\right]}{\overline{G}(\tau)}. \tag{3.6}$$

If $Y_1 < \tau,\ Y_2 < \tau, \ldots,\ Y_{n-1} < \tau,\ Y_n \geq \tau$.

Proof. Now if $N(t) = k$, and $X_1 < \ldots < X_k$ are the times of the minimal repairs. Then

$$p[N(t) = k;\ Y \geq t] = \overline{F}(t) \frac{1}{k!} \left[\int_0^t \overline{p}(x)\lambda(x)\,dx\right]^k, \tag{3.7}$$

where $\overline{p}(X_i)$ denote the probability of i-th minimal repair. Hence

$$E\left[\prod_{i=1}^{N(t)} \overline{p}(X_i)\right] = \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right].$$

And

$$\sum_{k=1}^{+\infty} E\left[\prod_{i=1}^{k} \overline{p}(X_i)\right] = \int_0^\tau \exp\left[-\int_0^t p(x)\lambda(x)\right]\lambda(t)\,dt - G(\tau).$$

Also

$$\int_0^\tau \overline{G}(t)\,dt = \int_0^\tau \overline{F}(t)\,dt$$
$$+\int_0^\tau \overline{F}(t|X_k)\left[\int_0^\tau \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right]\lambda(t)\,dt - G(\tau)\right]dt. \tag{3.9}$$

Let $F(t)$ be NBUE. From (3.9) we obtain

$$\int_0^\tau \overline{G}(t)\,dt \le \int_0^\tau \overline{F}(t)\,dt\left[\overline{G}(\tau) + \int_0^\tau \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right]\lambda(t)\,dt\right].$$

Hence $E[W]$ of the expected total time of an unit until planned replacement is bounded above by

$$\frac{\int_0^\tau \overline{F}(t)\,dt\left[\overline{G}(\tau) + \int_0^\tau \exp\left[-\int_0^t p(x)\lambda(x)\,dx\right]\lambda(t)\,dt\right]}{\overline{G}(\tau)}. \tag{3.10}$$

PROPOSITION 3.4. *Let X_i have IFR distribution F_i and Y_i have exponential distribution G_i and $E[X_i] = E[Y_i] = \mu_i$ for all $i = 1, \dots, n$. Let $\Phi(\cdot)$ is increasing functions on $[0, +\infty)$. Then*

$$E[\Phi \min \mathbf{X}] \ge E[\Phi \min \mathbf{Y}]. \tag{3.11}$$

Proof. Suppose Φ is increasing, and X_i is not identically equal to Y_i. Because X_i have IFR distribution F_i and Y_i have exponential distribution G_i with the same mean. Let X have IFR distribution F and there exists an exponential random variable W

with distribution function G, such that $E[X] = E[W]$. If $W(Y)$ have exponential distribution $F(G)$ with mean $E[W] > E[Y]$, that is, $\overline{F}_W(x) = \exp(-\gamma x)$ and $\overline{F}_Y(x) = \exp(-\delta x)$. The density of Y crosses W exactly once from above at, call x_0, that is, $f_W(x_0) = f_Y(x_0)$. Solving $\delta \exp(-\delta x_0) = \gamma \exp(-\gamma x_0)$ yields

$$x_0 = \frac{1}{\delta - \gamma} \ln\left(\frac{\delta}{\gamma}\right) > 0.$$

Let $X = \min \mathbf{X}$ and $Y = \min \mathbf{Y}$. By the Lemma 1.3 we have $E[\Phi X] \geq E[\Phi Y]$, and by Theorem 2.1 we have $E[W] \geq E[Y]$. Note that Y is also exponential. If $E[W] > E[Y]$, there exists a unique point x_0 where the density of Y crosses density of X and the crossing is from above. Since

$$E[\Phi(W)] = \int_0^{+\infty} \Phi(x)\gamma \exp(-\gamma x)\, dx,$$

and

$$E[\Phi(Y)] = \int_0^{+\infty} \Phi(x)\delta \exp(-\delta x)\, dx.$$

We get

$$\begin{aligned} &E[\Phi(W)] - E[\Phi(Y)] \\ &= \int_0^{x_0} [\Phi(x) - \Phi(x_0)][\gamma \exp(-\gamma x) - \delta \exp(-\delta x)]\, dt \\ &+ \int_{x_0}^{+\infty} [\Phi(x) - \Phi(x_0)][\gamma \exp(-\gamma x) - \delta \exp(-\delta x)]\, dx \leq 0, \end{aligned}$$

is nonnegative, thus $\Phi(x) \leq \Phi(x_0)$ and $\gamma \exp(-\gamma x) \leq \delta \exp(-\delta x)$ for $0 \leq x < x_0$. Similarly, $\Phi(x) \geq \Phi(x_0)$ and $\gamma \exp(-\gamma x) \geq \delta \exp(-\delta x)$ for $x_0 \leq x < +\infty$.

Model 3.2. *We consider a failure process with unit having life distribution F. Let $\gamma_{p(t)}(t)$ be the time from t until the next perfect repair or minimal repair, let $\delta_{p(t)}(t)$ be the time from the last perfect repair until t and let $\mu_{p(t)}(t)$ be the failure rate of $\gamma_{p(t)}(t)$.*

PROPOSITION 3.5. *Let F be IFR and let $\gamma_{p_1(t)}(t)$ and $\gamma_{p_2(t)}(t)$ are independent and let $\delta_{p_1(t)}(t)$ and $\delta_{p_2(t)}(t)$ are independent. If $p_1(t) \geq p_2(t)$, for all $t \geq 0$, then*

$$\gamma_{p_1(t)}(t) \overset{st}{\geq} \gamma_{p_2(t)}(t) \tag{3.13}$$

Proof. We first assumed that $\overline{G}_1(t) = \exp\left[\int_0^t p_1(x)\lambda(x)\,dx\right]$ and $\overline{G}_2(t) = \exp\left[\int_0^t p_2(x)\lambda(x)\,dx\right]$ for all $t > 0$, respectively. Let F be IFR and $p(t)$ be increasing. Note that

$$p[\gamma_{p(t)}(t) > x] = \int_0^{+\infty} p[\gamma_{p(t)}(t) > x|\delta_{p(t)}(t) = y]\,dp[\delta_{p(t)}(t) \leq y], \tag{3.14}$$

and also $p[\delta_{p(t)}(t) \leq y]$'s asymptotic distribution is

$$\lim_{t\to+\infty} p[\delta_{p(t)}(t) \leq y] = \frac{\int_0^y \overline{G}(z)\,dz}{\int_0^{+\infty} \overline{G}(x)\,dx}.$$

From Theorem 3 of Brown[browna], that $\gamma_{p(t)}(t)$ and $\delta_{p(t)}(t)$ are stochastically decreasing in t.

$$\begin{aligned} p[\gamma_{p_1(t)}(t) > x] &\geq \frac{\int_0^{+\infty} \overline{F}(x|y)\overline{G}_2(y)\,dy}{\int_0^{+\infty} \overline{G}_2(x)\,dx} \\ &= p[\gamma_{p_2(t)}(t) > x], \end{aligned}$$

where $\int_0^{+\infty} \overline{F}(x|y)\,dy$ is decreasing in y, and $\overline{G}(y)/\int_0^{+\infty} \overline{G}(x)\,dx$ is stochastically decreasing in y. Hence $p[\gamma_{p(t)}(t) > x]$ is decreasing in t. Furthermore, the failure rate function belonging to $\gamma_{p(t)}(t)$ is

$$\mu_{p(t)}(t) = \frac{\int_0^{+\infty} \overline{F}(x|y)\overline{G}(y)\lambda(x+y)\,dy}{\int_0^{+\infty} \overline{F}(x|y)\overline{G}(y)\,dy}. \tag{3.16}$$

If $p_1(t) \geq p_2(t)$, let $\mu_{p(t)}$ be the failure rate function belonging to $\gamma_{p(t)}(t)$, since

$$\mu_{p_1(t)}(t) \leq \mu_{p_2(t)}(t). \tag{3.17}$$

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