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TOPICS ON FUNDAMENTAL TOPOLOGICAL ALGEBRAS

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Abstract. A class of topological algebras,which we call it a fundamental one, has already been introduced to generalize the locally bounded and locally convex algebras. To prove the basic theorems on fundamental algebras, the first successful step is the new version of the Cohen factorization theorem. Here we recall it and prove some new results on fundamental topological algebras.

1. Introduction

In [1], the author introduces a new class of topological algebras (namely fundamental one), to generalize the meaning of locally bounded and locally convex algebras. A natural question is to ask for generalizing the basic results on this new class. This is of course a wide question and the first successful step answering it, has already been down in [1]. In fact, we have proved there the most famous Cohen factorization theorem for complete metrizable fundamental topological algebras [1,Theorem,4.1] .

In this note, we give an example to exhibit the ability of the main theorem in [1],i.e., 4.1 and,then we generalize some basic results to fundamental topological algebras. At first, in section 2, we recall some definitions and related results.

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2. Definitions and related results

DEFINITION 2.1. A topological linear space A is said to be fundamental one if there exists $b > 1$ such that for every sequence (x_n) of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that (x_n) is cauchy.

DEFINITION 2.2. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental.

PROPOSITION 2.3. *Let A be a fundamental topological linear space. Then, for all $c > 1$ and every sequence (x_n) of A, the convergence of $c^n(x_n - x_{n-1}) \to 0$ in A implies that (x_n) is a cauchy sequence.*

3. An application of fundamental topological algebras

When A is a (non locally bounded) locally convex F-algebra, and B is a (non locally convex) locally bounded F-algebra, then, $A \bigoplus B$ with product topology and componentwise definitions for addition, scaler product, and multiplication is a fundamental F-algebra, and if the Cohen theorem could be proved for A and B then we can prove it for $A \bigoplus B$. Since $A \bigoplus B$ is neither locally bounded nor locally convex, if we change the definition of product on $A \bigoplus B$, we may not be able to factor it by the previous versions of the Cohen factorization theorem.

Here, we prove a theorem and give an example to apply the Theorem 4.1 of [1].

THEOREM 3.1. *There exists a fundamental F-algebra for which the Cohen factorization theorem holds but it can not be proved by the previous well-known theorems.*

Proof. Let A be a locally bounded, but not locally convex, F-algebra with a bounded left approximate identity $(e_\lambda)_{\lambda \in \Lambda}$. Let also X be a locally convex,but not locally bounded F-algebra which is a two-sided Frechet A-module with $e_\lambda x - x \to 0$ in X. Let moreover for each s, t, u in A or X, $s(tu) = (st)u$. Define $Z =$

$X \bigoplus A$ to be the direct sum of X and A with product topology. Then Z is a fundamental topological linear space which is not locally bounded and not locally convex.

For $(x, a) \in Z$ and $(y, b) \in Z$, we define the multiplication as:

$$(x, a).(y, b) = (xy + xb + ay, ab) \in Z.$$

Now, let $U \times V$ be a basic neighborhood for zero in Z. Choose the neighborhood W of zero in X such that $W + W + W \subseteq U$, and the neighborhood V_0 and U_0 of zeros respectively in A and X such that:

$$U_0 V_0 \subseteq W, \ V_0 U_0 \subseteq W, \ U_0^2 \subseteq W, \ V_0^2 \subseteq V,$$

and let $(x, a) \in U_0 \times V_0$ and $(y, b) \in U_0 \times V_0$, then

$$xy \in U_0^2 \subseteq W, xb \in U_0 V_0 \subseteq W, ay \in V_0 U_0 \subseteq W and, ab \in V_0^2 \subseteq V$$

thus $(x, a).(y, b) \in U \times V$. Therefore the multiplication is jointly continuous.

Now let $(x, a) \in Z$ and $U \times V$ be a neighborhood for zero in Z. Take $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies $e_\lambda x - x \in U$ and $e_\lambda a - a \in V$.Then

$$(0, e_\lambda).(x, a) - (x, a) \in U \times V,$$

i.e.$((0, e_\lambda))_{\lambda \in \Lambda}$ is a left approximate identity for Z. Since $(e_\lambda)_{\lambda \in \Lambda}$ is uniformly bounded in $A, \exists k > 0$ such that the set $\{k^{-m} {e_\lambda}^m : \lambda \in \Lambda, m \in Z^+\}$ is bounded in A. Therefore for $U \times V$ we take $M > 0$ such that for all $m \in Z^+$ and all $\lambda \in \Lambda, M^{-1} k^{-m} e_\lambda^m \in V$ and then

$$M^{-1} k^{-m} (0, \lambda)^m = (0, M^{-1} k^{-m} e_\lambda^m) \in U \times V,$$

i.e.$((0, e_\lambda))_{\lambda \in \Lambda}$ is uniformly bounded in Z. Now,since Z is a fundamental Frechet algebra with uniformly bounded left approximate identity, each $(x, a) \in Z$ can be factored as

$$(x, a) = (y, b).(z, c)[1, Theorem 4.1],$$

where by the previous factorization theorem we can not factor it.

EXAMPLE. Let $0 < p < 1$ and $T = \{t_1, t_2, \cdots\}$ be a set of symbols and S be the semigroup generated by T with relation $t_i t_j = t_{min(i,j)}$ if $i \neq j$ and $t_i^n t_i = t_i^{n+1}$. Then $S = \{t_i^j : i, j \in Z^+\}$.

Let $A = \{\sum_{i,j=1}^{\infty} \alpha_{ij} t_i^j : \sum_{i,j} \mid \alpha_{ij} \mid^p (j+1)^p < \infty\}$ with $\| \sum_{i,j} \alpha_{ij} t_i^j \|_p = \sum_{i,j} \mid \alpha_{ij} \mid^p (j+1)^p$. Then A is locally bounded,but not locally convex,F-algebra generated by S. It is clear that $(t_n)_n$ is a bounded sequential approximate identity with $\| t_n \|_p = 2^p$. Let also

$$X = \{\sum_{i,j}^{\infty} \alpha_{ij} t_i^j : \sum_{i,j} \mid \alpha_{ij} \mid p_m(t_i^j) < \infty, m = 1, 2, ...\},$$

where

$$p_m(t_i^j) = \begin{cases} (j+1)^p & \text{if } i \leq m \\ 1 & \text{if } i > m. \end{cases}$$

and $P_m(\sum_{i,j} \alpha_{ij} t_i^j) = \sum_{i,j} \mid \alpha_{ij} \mid p_m(t_i^j)$. Then each p_m is a sub-multiplicative seminorm and X is an LMC Frechet algebra, such that for each $M > 0$ and $m \in Z^+$,

$$\{x \in X : p_m(x) < 1\} \not\subseteq M\{x : p_{m+1}(x) < 1\},$$

and therefore X is not locally bounded.

Suppose $a = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} t_i^j \in A$, then $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mid \alpha_{ij} \mid^p (j+1)^p < \infty$. Therefore $\sum_{i=1}^{\infty} (\sum_{j=1}^{\infty} \mid \alpha_{ij} \mid^p) < \infty$ and $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \mid \alpha_{ij} \mid^p < \infty$, hence $\exists N_1$ such that for all $i \geq N_1, \sum_{J=1}^{\infty} \mid \alpha_{ij} \mid^p < 1$. Thus, for $i \geq N_1$ and $j \in Z^+, \mid \alpha_{ij} \mid^p \leq \sum_{j=1}^{\infty} \mid \alpha_{ij} \mid^p < 1$. Also,$\exists N_2$ such that for all $j \geq N_2$ and $i \in Z^+, \mid \alpha_{ij} \mid^p < 1$. Take $N_0 = max(N_1, N_2)$. Now ,if $i \geq N_0$ or $j \geq N_0$ we have $\mid \alpha_{ij} \mid < 1$,therefore $\mid \alpha_{ij} \mid \leq \mid \alpha_{ij} \mid^p$ for $i \geq N_0$ or $j \geq N_0$.

Now, for each $m \in Z^+$,

$$\sum_{i,j} \mid \alpha_{ij} \mid p_m(t_i^j) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \mid \alpha_{ij} \mid (j+1)^p$$

$$= \sum_{i=1}^{N_0} \sum_{j=1}^{N_0} \mid \alpha_{ij} \mid (j+1)^P + \sum_{N_0+1}^{\infty} \sum_{j=1}^{N_0} \mid \alpha_{ij} \mid (j+1)^p$$

$$+\sum_{i=1}^{N_0}\sum_{j=N_0+1}^{\infty} | \alpha_{ij} | (j+1)^p + \sum_{i=N_0+1}^{\infty}\sum_{j=N_0+1}^{\infty} | \alpha_{ij} | (j+1)^p$$

$$\leq \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} | \alpha_{ij} | (j+1)^p + \sum_{i=N_0+1}^{\infty}\sum_{j=1}^{\infty} | \alpha_{ij} |^p (j+1)^p$$

$$+\sum_{i=1}^{N_0}\sum_{j=N_0+1}^{\infty} | \alpha_{ij} |^p (j+1)^p + \sum_{i,j=N_0+1}^{\infty} | \alpha_{ij} |^p (j+1)^p$$

$$\leq \sum_{i=1}^{N_0}\sum_{j=1}^{N_0} | \alpha_{ij} | (j+1)^p + 3 \| a \| < \infty,$$

and so $a \in X$, i.e. $A \subseteq X$.

Now,for $a \in A$ and $x \in X$ we define $a.x = ax$ in the usual way. Then X is a two-sided A-module. Let $\epsilon > 0, j \in Z^+$ and $U = \{x \in X : p_j(x) < \epsilon\}$ then,there exists $m \in Z^+$ and $0 < \delta < 1$ with

$$\{x : p_m(x) < \delta\}.\{x : p_m(x) < \delta\} \subseteq U.$$

Let $V = \{a \in A : \| a \|_p < \delta\}$ and take $a = \sum \alpha_{ij} t_i^j \in V$. Since, for each $i, j, | \alpha_{ij} | \leq \| a \|_p < \delta < 1$, we have $p_m(a) < \delta$. Thus $V \subseteq \{x : p_m(x) < \delta\}$ and so $V \times \{x \in X : p_m(x) < \delta\} \subseteq U$, i.e. X is a topological A-module.Since for each $x \in X, t_n x - x \to 0$ in X,thus X is an essential Frechet A- module(left and right).

Now, by the Theorem 3.1,the Cohen factorization theorem holds for $Z = X \bigoplus A$, where by the previous well-known theorems we can not prove it.

4. Some new results on fundamental topological algebras

In this section we state and prove some basic results on fundamental topological algebras.

THEOREM 4.1. *Let A be a complete metrizable fundamental topological algebra, and $x \in A$. If for some $b > 1, b^n x^n \to 0$ in A, then:*

a) x is quasi-invertible and $x^0 = -\sum_{n=1}^{\infty} x^n$,
b) If A possesses a unit element,then $1 - x$ is invertible and

$$(1-x)^{-1} = 1 + \sum_{n=1}^{\infty} x^n.$$

Proof. Suppose $s_n = \sum_{k=1}^{n} x^n$. Since $b^n(s_n - s_{n-1}) = b^n x^n$, (s_n) is a cauchy sequence. Let $y = \sum_{k=1}^{\infty} x^k = lim_{n\to\infty} s_n$. Now

$$(1-x)(1+s_n) = (1-x) + \sum_{k=1}^{n}(x^k - x^{k+1}) = (1-x) + x - x^{n+1} \to 1.$$

Since the multiplication is continuous on A, we have

$$(1-x)(1+y) = (1+y)(1-x) = 1.$$

COROLLARY. *Let A be with the conditions of the theorem* 4.1 *and $b > 1$. If $b^n(1-x)^n \to 0$ in A then x is invertible.*

Now, we have some results in a class of fundamental topological algebras. Although every locally bounded topological algebra satisfies in the condition of this class, but until know the author has no example of a locally convex one.

DEFINITION 4.2. A fundamental topological algebra is called to be locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. We call such an algebra, an FILM algebra.

THEOREM 4.3. *Let A be a complete metrizable FLM algebra with a unit element. Then the set of all invertible elements of A, is an open subset of A.*

Proof. Let $x_0 \in Inv(A)$ and U_0 be a neighborhood of zero which satisfies in 4.2, and $b > 1$. Choose a neighborhood V_0 such that $x_0^{-1}V_0 \subseteq b^{-1}U_0$. Now let $y \in x_0 - V_0$. Then $x_0 - y \in V_0$ and $1 - x_0^{-1}y \in x_0^{-1}V_0$. Since $b(1 - x_0^{-1}y) \in bx_0^{-1}V_0 \subseteq U_0$, we have $b^n(1 - x_0^{-1}y)^n \in U_0^n$ and $b^n(1 - x_0^{-1}y)^n \to 0$ and so, $x_0^{-1}y \in Inv(A)$. Since $y = x_0(x_0^{-1}y)$ thus $y \in Inv(A)$.

THEOREM 4.4. *Let A be a complete metrizable FLM algebra with a unit element and let $a \in A$. Then, the $sp(a)$ is compact.*

Proof. Let $a \in A$. Define $\phi : C \to A$ by $\phi(\lambda) = \lambda - a$ then ϕ is continuous and $\phi^{-1}(Inv(A))$ is open. since $C \setminus \phi^{-1}(Inv(A)) = \phi^{-1}(Sing(A)) = sp(a)$ so $sp(a)$ is closed.

Suppose U_0 be a neighborhood of zero which satisfies in 4.2, and let $\alpha \in C \setminus \{0\}$ such that $a \in \alpha U_0$, then $(\alpha^{-1}a)^n \to 0$. Suppose $|\lambda| > 1$ and take b such that $|\lambda| > b > 1$, then $(\frac{b}{\lambda})^n \to 0$ and $b^n(\frac{a}{\alpha\lambda})^n \to 0$. Therefore $1 - \frac{a}{\alpha\lambda} \in Inv(A)$ and $\alpha\lambda \notin sp(a)$. Now,if $\beta \in sp(a)$, put $\lambda = \frac{\beta}{\alpha}$ and then, $|\lambda| \leq 1$ i.e. $|\beta| \leq |\alpha|$, thus the $sp(a)$ is bounded.

THEOREM 4.5. *Let A be a complete metrizable FLM algebra. Then, every multiplicative linear functional is continuous.*

Proof. Let $\phi : A \to C$ be a non-zero multiplicative linear functional and $b > 1$. Suppose $x \in A$ with $b^n x^n \to 0$. Put $S_n = \sum_{k=1}^{n} x^k$, then $y = \sum_{k=1}^{\infty} x^k \in A$. Since,

$$y - xy = limS_n - xlimS_n = lim(S_n - xS_n) = x,$$

we have $\phi(x) \neq 1$. If $|\phi(x)| > 1$, take $x_0 = \frac{x}{\phi(x)}$. Since

$$b^n x_0^n = b^n \frac{x^n}{\phi(x)^n} = \frac{1}{\phi(x)^n} b^n x^n \to 0,$$

thus we must have $\phi(x_0) \neq 1$, which is impossible; therefore if $b^n x^n \to 0$ then $|\phi(x)| < 1$.

Now, let (x_n) be any null sequence in A and let U_0 satisfies in 4.2. For $\epsilon > 0$, take $n \in Z^+$ so large that $b^{-1}\epsilon x_n \in U_0$. Fix this n and suppose V be any neighborhood of zero. There exists $K_0 \in Z^+$ such that $k \geq K_0$ implies that $U_0^k \subseteq V$ and so $b^k(\epsilon^{-k} x_n^k) \in V$ thus $lim_{k\to\infty} b^k(\epsilon^{-1}x_n)^k = 0$ and so, $|\phi(\epsilon^{-1}x_n)| < 1$; i.e. $|\phi(x_n)| < \epsilon$, which says $lim_{x\to 0}\phi(x) = 0$.

THEOREM 4.6. *Let A be a complete metrizable FLM algebra with unit element, and ϕ a linear functional on A such that $\phi(1) = 1$ and, $ker\phi \subseteq Sing(A)$. Then ϕ is continuous.*

Proof. Let $b > 1$ and $x \in A$ such that $b^n x^n \to 0$ then $1 - x \notin Sing(A)$ and then $\phi(x) \neq \phi(1) = 1$. If $| \phi(x) |> 1$, take $x_0 = \frac{1}{\phi(x)} x$. Since

$$b^n x_0^n = (\frac{1}{\phi(x)})^n b^n x^n \to 0,$$

so $1 - x \notin Sing(A)$ and thus $\phi(x_0) \neq 1$ which is impossible.

Now suppose U_0 be as in $4.2, x_n \to 0$; and $\epsilon > 0$. There exists $N \in Z^+$ such that $n \geq N$ implies that $b\epsilon^{-1} x_n \in U_0$ and hence, $lim_{k\to\infty} b^k (\epsilon^{-k} {x_n}^k) = 0$. Therefore $| \phi(\epsilon^{-1} x_n |< 1$, i.e. ϕ is continuous.

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