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SEMI–INVARIANT SUBMANIFOLDS OF CODIMENSION 3 SATISFYING $\mathfrak{L}_\xi \nabla = 0$ IN A NONFLAT COMPLEX SPACE FORM

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Abstract. In this paper, we characterize some semi-invariant submanifolds of codimension 3 with almost contact metric structure (ϕ, ξ, g) satisfying $\mathfrak{L}_\xi \nabla = 0$ in a nonflat complex space form, where ∇ denotes the Riemannian connection induced on the submanifold, and $\mathfrak{L}_\xi$ is the operator of the Lie derivative with respect to the structure vector field ξ.

0. Introduction

A submanifold M is called a *semi-invariant submanifold* of a Kaehlerian manifold $\tilde{M}$ with complex structure if there exists a differentiable distribution $\Delta : p \to \Delta_p \subset M_p$ on M such that Δ is J-invariant and the complementary orthogonal distribution $\Delta^\perp$ is totally real and $\dim\Delta^\perp = 1$, where M_p denote the tangent space at each point p in M ([2], [14], [16]). In this case, M admits an induced almost contact metric structure (ϕ, ξ, g). A typical example of a semi-invariant submanifold is real hypersurfaces. Furthermore, new examples of nontrivial semi-invariant submanifolds in a complex projective space $\mathbb{C}P^n$ are constructed

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in [7] and [15]. Therefore we may except to generalize some results which are valid in a real hypersurface to a semi-invariant submanifold.

For the real hypersurface case, when $\tilde{M}$ is a nonflat complex space form, many results are known. Two of them, Okumura ([11]) and Montiel-Romero ([9]) characterized real hypersurfaces of type A_1 and A_2 in a complex space form $M_n(c), c \neq 0$ by the property that the shape operator A and structure tensor field ϕ commute. Namely , they proved the followings respectively

THEOREM O ([11]). *Let M be a connected real hypersurface of a complex projective space $\mathbb{C}P^n$. If M satisfies $\phi A = A\phi$, then M is locally congruent to one of the following spaces :*

(A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane $\mathbb{C}P^{n-1}$, where $0 < r < \frac{\pi}{2}$),*

(A_2) *a tube of radius r over a totally geodesic $\mathbb{C}P^k$, $(1 \leq k \leq n-2)$, where $0 < r < \pi/2$.*

THEOREM MR ([9]). *Let M be a connected real hypersurface of a complex hyperbolic space $\mathbb{C}H^n$. If M satisfies $\phi A = A\phi$, then M is locally congruent to one of the following spaces :*

(A_0) *a Montiel tube,*

(A_1) *a tube of a totally geodesic hyperplane $\mathbb{C}H^k$ ($k = 0$ or $n-1$),*

(A_2) *a tube of totally geodesic $\mathbb{C}H^k$, $(1 \leq k \leq n-2)$.*

We denote by ∇ the Levi-Civita connection with respect to induced Riemannian metric tensor g on M. Then, it is proved in [3] that another characterization of real hypersurfaces of type A_1 and A_2 in a nonflat complex space form is given. More specifically, Choe and Lee proved the following

THEOREM CL ([3]). *Let M be a connected real hypersurface of a nonflat complex space form. If M satisfies $\mathfrak{L}_\xi \nabla = 0$, then M is of type A_0, A_1 or A_2, where $\mathfrak{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector ξ.*

On the other hand, semi-invariant submanifolds of codimension 3 in a complex projective space $\mathbb{C}P^{n+1}$ have been studied in [5], [6],[7],[18] and so on by using properties of induced almost contact metric structure and those of the third fundamental form of the submanifold.

The main purpose of the present paper is to extend Theorem CL under certain conditions on a semi-invariant submanifold of codimension 3 in a nonflat complex space form.

All manifolds in this paper are assumed to be connected and of class C^∞, and the dimension of submanifold is greater than 2.

1. Preliminaries

At first we review fundamental properties on a semi-invariant submanifold of a complex space form. Let $\tilde{M}$ be a real $2(n+1)$-dimensional Kaehlerian manifold equipped with parallel almost complex structure J and a Riemannian metric tensor G and covered by a system of coordinate neighborhoods $\{\tilde{V}; y^A\}$.

Let M be a real (2n-1)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; x^h\}$ and immersed isometrically in $\tilde{M}$ by the immersion $i : M \to \tilde{M}$. We represent the immersion i locally by $y^A = y^A(x^h)$ and $B_j = (B_j{}^A)$ are $(2n-1)$-linealy independent local tangent vectors of M, where $B_j{}^A = \partial_j y^A$ and $\partial_j = \partial/\partial x^j$. Where here and in the sequel, indices $A, B, \cdots$ run over $1, 2, \cdots, 2(n+1)$ and $i, j, \cdots$ run from $1, 2, \cdots$ to $2n-1$. The summation convention will be used with respect to those system of indices. Three mutually orthogonal unit normals C, D and E may be chosen. Since the immersion i is isometric, the induced Riemannian metric tensor g with components g_{ji} on M is given by $g_{ji} = G(B_j, B_i)$.

Denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g, equations of the Gauss for M of $\tilde{M}$ is obtained :

$$\nabla_j B_i = A_{ji} C + K_{ji} D + L_{ji} E, \tag{1.1}$$

where A_{ji}, K_{ji} and L_{ji} are components of the second fundamental forms in the direction C, D and E respectively. Equations of

Weingarten are also given by

$$
\begin{aligned}
\nabla_j C &= -A_j{}^h B_h + l_j D + m_j E, \\
\nabla_j D &= -K_j{}^h B_h - l_j C + n_j E, \qquad (1.2)\\
\nabla_j E &= -L_j{}^h B_h - m_j C - n_j D,
\end{aligned}
$$

where $A = (A_j{}^h), A_{(2)} = (K_j{}^h)$ and $A_{(3)} = (L_j{}^h)$, which are related by $A_{ji} = A_j{}^r g_{ir}, K_{ji} = K_j{}^r g_{ir}$ and $L_{ji} = L_j{}^r g_{ir}$ respectively, and l_j, m_j and n_j being components of the third fundamental forms.

As is well-known, a submanifold M of a Kaehlerian manifold $\tilde{M}$ is said to be a CR *submanifold* ([1], [19]) if it is endowed with a pair of mutually orthogonal complementary differentiable distribution $(\Delta, \Delta^{\perp})$ such that for any $p \in M$ we have $J\Delta_p = M_p, J\Delta_p{}^{\perp} \subset M_p{}^{\perp}$, where $M_p{}^{\perp}$ denotes the normal space of M at p. In particular, M is said to be *a semi-invariant submanifold* if dim $dim\Delta^{\perp} = 1$, and the unit normal vector in $J\Delta^{\perp}$ is called a *distinguished normal* to the submanifold and denoted this by C ([2], [16]). Then we can write

$$
JB_i = \phi_i{}^h B_h + \xi_i C, \quad JC = -\xi^h B_h, \quad JD = -E, \quad JE = D, \qquad (1.3)
$$

where we have put $\phi_{ji} = G(JB_j, B_i), \xi_j = G(JB_j, C), \xi^h$ being associated components of ξ_h ([7]). A tensor field of type (1,1) with components $\phi_j{}^h$ will be denoted by ϕ. By properties of the almost complex structure J, it is, using (1.3), seen that

$$
\begin{aligned}
&\phi_i{}^r \phi_r{}^h = -\delta_i{}^h + \xi_i \xi^h, \quad \xi_r \phi_i{}^r = 0, \quad \xi^r \phi_r{}^h = 0, \\
&\xi_r \xi^r = 1, \quad g_{rs} \phi_j{}^r \phi_i{}^s = g_{ji} - \xi_j \xi_i.
\end{aligned}
$$

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla^{\perp} C$. The distinguished normal is said to be *parallel* in the normal bundle if we have $\nabla^{\perp} C = 0$, that is, l_j and m_j vanish identically.

Since J is parallel, differentiating (1.3) covariantly along M and making use of (1.1), (1.2) and (1.3) itself, we find ([18])

$$
\nabla_j \phi_i{}^h = -A_{ji} \xi^h + A_j{}^h \xi_i, \qquad (1.4)
$$

$$(1.5)\qquad \nabla_j \xi_i = -A_{jr}\phi_i{}^r,$$

$$(1.6)\qquad K_{ji} = -L_{jr}\phi_i{}^r - m_j\xi_i,$$

$$(1.7)\qquad L_{ji} = K_{jr}\phi_i{}^r + l_j\xi_i.$$

REMARK 1. To write our formulas in a convention form, in what follows we denote by

$$\alpha = A_{rs}\xi^r\xi^s, \quad \beta = A_{rs}^2\xi^r\xi^s, \quad h = T_rA, k = T_rA_{(2)},$$
$$h_{(2)} = T_rA^2, \quad K_{(2)} = T_rA_{(2)}^2, \quad L_{(2)} = T_rA_{(3)}^2,$$

and for a function f we denote by ∇f the gradient vector field of f.

We notice here that we may assume $T_rA_{(3)} = 0$ (see[7]). Thus, it is, using (1.6) and (1.7), verified that

$$(1.8)\qquad K_{jr}\xi^r = -m_j, \quad L_{jr}\xi^r = l_j,$$

$$(1.9)\qquad m_r\xi^r = -k, \quad l_r\xi^r = 0.$$

Further, we obtain

$$(1.10)\qquad \phi_{jr}m^r = -l_j, \quad \phi_{jr}l^r = m_j + k\xi_j,$$

$$(1.11)\qquad K_{jr}L_i{}^r + K_{ir}L_j{}^r + l_jm_i + l_im_j = 0.$$

Now, we put $U = \nabla_\xi\xi$. Then U is orthogonal to ξ. Thus we have

$$(1.12)\qquad \phi_{jr}U^r = A_{jr}\xi^r - \alpha\xi_j,$$

$$U^r\nabla_j\xi_r = A_{jr}^2\xi^r - \alpha A_{jr}\xi^r \tag{1.13}$$

because of (1.5). From (1.12) we get $g(U,U) = \beta - \alpha^2$. Therefore we easily see that $A\xi = \alpha\xi$ if and only if $\beta - \alpha^2 = 0$. Differentiating (1.12) covariantly and taking account of (1.4) and (1.5), we find

(1.14)
$$\xi_j(A_{kr}U^r + \alpha_k) + \phi_{jr}\nabla_k U^r = \xi^r\nabla_k A_{jr} - A_{jr}A_{ks}\phi^{rs} + \alpha A_{kr}\phi_j{}^r,$$

where we put $\alpha_k = \nabla_k\alpha$.

The ambient Kaehlerian manifold $\tilde{M}$ is assumed to be of constant holomorphic sectional curvature c, which is called a complex space form and denoted $M_{n+1}(c)$. Then equations of the Gauss and Codazzi are given by

(1.15)
$$\begin{aligned} R_{kjih} = {} & \frac{c}{4}(g_{kh}g_{ji} - g_{jh}g_{ki} + \phi_{kh}\phi_{ji} - \phi_{jh}\phi_{ki} - 2\phi_{kj}\phi_{ih}) \\ & + A_{kh}A_{ji} - A_{jh}A_{ki} + K_{kh}K_{ji} - K_{jh}K_{ki} + L_{kh}L_{ji} - L_{jh}L_{ki}, \end{aligned}$$

$$\begin{aligned} & \nabla_k A_{ji} - \nabla_j A_{ki} - l_k K_{ji} + l_j K_{ki} - m_k L_{ji} + m_j L_{ki} \\ & = \frac{c}{4}(\xi_k\phi_{ji} - \xi_j\phi_{ki} - 2\xi_i\phi_{kj}), \end{aligned} \tag{1.16}$$

$$\nabla_k K_{ji} - \nabla_j K_{ki} = l_j A_{ki} - l_k A_{ji} + n_k L_{ji} - n_j L_{ki}, \tag{1.17}$$

$$\nabla_k L_{ji} - \nabla_j L_{ki} = m_j A_{ki} - m_k A_{ji} - n_k K_{ji} + n_j K_{ki}, \tag{1.18}$$

where R_{kjih} are covariant components of the Riemann-Christoffel curvature tensor of M, and those of the Ricci by

$$\nabla_k l_j - \nabla_j l_k = A_{jr}K_k{}^r - A_{kr}K_j{}^r + m_j n_k - m_k n_j, \tag{1.19}$$

$$\nabla_k m_j - \nabla_j m_k = A_{jr}L_k{}^r - A_{kr}L_j{}^r + n_j l_k - n_k l_j, \tag{1.20}$$

$$\nabla_k n_j - \nabla_j n_k = K_{jr}L_k{}^r - K_{kr}L_j{}^r + l_j m_k - l_k m_j + \frac{c}{2}\phi_{kj}. \tag{1.21}$$

2. Semi-invariant submanifolds satisfying $\gamma_\xi \nabla = 0$

First of all, we prove

LEMMA 2.1. *Let M be a semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$. Then we have*

$$divU = \frac{1}{2}\|A\phi - \phi A\|^2 - h_{(2)} + \alpha h + \frac{c}{2}(n-1) - 2l_r l^r,$$

$\|X\|$ means the usual norm for any vector field X on M.

Proof. From (1.4) and (1.5), we have

$$\nabla_i \nabla_j \xi^i = hA_{jr}\xi^r - A_{jr}^2 \xi^r - (\nabla_k A_{jr})\phi^{kr}.$$

which together with (1.8), (1.10) and (1.15) implies that

$$\xi^j \nabla_i \nabla_j \xi^i = h\alpha - \beta + \frac{c}{2}(n-1) - l_r l^r - m_r m^r + k^2. \tag{2.1}$$

On the other hand, transforming (1.7) by $L_k{}^i$ and using (1.6) and (1.8), we get

$$L_{jk}^2 - K_{jk}^2 = l_j l_k - m_j m_k, \tag{2.2}$$

which connected with (1.8) and (1.9) gives

$$l_r l^r = m_r m^r - k^2. \tag{2.3}$$

Thus (2.1) turns out to be

$$\xi^j \nabla_i \nabla_j \xi^i = h\alpha - \beta + \frac{c}{2}(n-1) - 2l_r l^r.$$

Since we have $divU = (\nabla_j \xi_i)(\nabla^i \xi^j) + \xi^j \nabla_i \nabla_j \xi^i$, the above equation implies

$$\|A\phi - \phi A\|^2 = 2divU + 2\{h_{(2)} - \alpha h - \frac{c}{2}(n-1) + 2l_r l^r\},$$

where we have used (1.5). This completes the proof.

In the rest of this paper we shall suppose that M is a real $(2n-1)$-dimensional semi-invariant submanifold of codimension 3 in a complex space form $M_{n+1}(c)$. Suppose that $\mathfrak{L}_\xi \nabla = 0$, that is,

$$\mathfrak{L}_\xi \left\{ {h \atop j\ i} \right. = 0 \quad \text{where} \quad \left\{ {h \atop j\ i} \right.$$

is the Christoffel symbols formed with g_{ji}. Then we have (cf.[17])

$$\nabla_j \nabla_i \xi^h + R_{kji}{}^h \xi^k = 0.$$

Thus, if we use (1.4), (1.5), (1.8) and (1.15), then we obtain

$$\begin{aligned}(\nabla_j A_{ir})\phi^{hr} &= \frac{c}{4}(g_{ji}\xi^h - \xi_i \delta_j^h) + (A_r{}^h \xi^r)A_{ji} - \xi^h A_{ji}^2 \\ &\quad + m_i K_j{}^h - m^h K_{ji} + l^h L_{ji} - l_i L_j{}^h,\end{aligned} \tag{2.4}$$

which together with (1.8) and (1.9) yields

$$\alpha A_{ji} - A_{ji}^2 + \frac{c}{4}(g_{ji} - \xi_j \xi_i) + kK_{ji} - m_j m_i + l_j l_i = 0. \tag{2.5}$$

If we multiply $\xi^j \xi^i$ to this and sum for j and i, and take account of (1.8) and (1.9), then we have $\beta = \alpha^2$ and hence $A\xi = \alpha\xi$. Applying (2.5) by g^{ji} and using (2.3), we find

$$\alpha h - h_{(2)} + \frac{c}{2}(n-1) - 2l_r l^r = 0.$$

From this and Lemma 2.1, it follows that

$$A\phi = \phi A \tag{2.6}$$

because of the fact that $A\xi = \alpha\xi$. Thus (1.14) is reduced to

$$\xi^r \nabla_k A_{jr} = A_{kr}^2 \phi_j{}^r - \alpha A_{kr} \phi_j{}^r + \alpha_k \xi_j,$$

from which, taking the skew-symmetric part and using (1.8), (1.16) and (2.6),

$$(A_{kr}^2 - \alpha A_{kr})\phi_j{}^r + \frac{1}{2}(\alpha_k \xi_j - \alpha_j \xi_k) = m_k l_j - m_j l_k - \frac{c}{4}\phi_{kj},$$

which together with (1.9) implies that

$$\alpha_j = x\xi_j + 2kl_j, \tag{2.7}$$

where we have put $x = \alpha_t\xi^t$. Combining the last two equations, it follows that

$$(A_{kr}^2 - \alpha A_{kr} - \frac{c}{4}g_{kr})\phi_j{}^r = m_k l_j - m_j l_k + k(\xi_k l_j - \xi_j l_k).$$

Transforming this by $\phi_i{}^j$ and making use of (1.10), we find

$$-A_{ki}^2 + \alpha A_{ki} + \frac{c}{4}(g_{ki} - \xi_k\xi_i) = l_k l_i + m_k m_i + k(m_k\xi_i + m_i\xi_k) + k^2\xi_k\xi_i,$$

which connected with (2.5) implies that

$$kK_{ji} + 2l_j l_i + k(m_j\xi_i + m_i\xi_j) + k^2\xi_j\xi_i = 0. \tag{2.8}$$

If we take the trace of the last equation, then we obtain $l_r l^r = 0$ and hence $l_j = 0$. Thus, the second equation of (1.10) turns out to be

$$m_j = -k\xi_j. \tag{2.9}$$

Therefore (2.8) implies that $k = 0$. Consequently the distinguished normal is parallel in the normal bundle. Thus we have

PROPOSITION 2.2. *Let M be a semi-invariant submanifold of codimension 3 in a nonflat complex space form $M_{n+1}(c), c \neq o$. If M satisfies $\mathfrak{L}_\xi\nabla = 0$, then we have $A\phi = \phi A$ and that the distinguished normal is parallel in the normal bundle, where $\mathfrak{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector ξ.*

From (1.15) we verify that the Ricci tensor S of M with component S_{ji} is given by

$$S_{ji} = \frac{c}{4}\{(2n+1)g_{ji} - 3\xi_j\xi_i\} + hA_{ji} - A_{ji}^2 - 2K_{ji}^2,$$

where we have used (1.11) and the fact that the distinguished normal is parallel in the normal bundle. Thus the scalar curvature ρ of M is given by

$$\rho = c(n^2-1) + h^2 - h_{(2)} - 2K_{(2)}.$$

If we supose that $\rho - c(n^2-1) - h^2 + h_{(2)} \geq 0$, then we have $K_{(2)} = 0$ and hence $A_{(2)} = A_{(3)} = 0$.

Let $N_0(p) = \{\eta \in M_p^{\perp} | A_\eta = 0\}$ and $H_0(p)$ the maximal J-invariant subspace of $N_0(p)$. Because we have $A_{(2)} = A_{(3)} = 0$ and $\nabla^{\perp} C = 0$, the orthogonal complement of $H_0(p)$ is invariant under parallel translation with respect to the normal connection. Consequently, by the reduction theorm in [4], [13] and by Proposition 2.2, we have

LEMMA 2.3. *Let M be a semi-invariant submanifold of codimension 3 in a nonflat complex space form. If M satisfies $\mathfrak{L}_\xi \nabla = 0$ and the scalar curvature ρ of M satisfies $\rho - c(n^2-1) - h^2 - h_{(2)} \geq 0$, then M is a real hypersurface with $A\phi = \phi A$ in $\mathbb{C}P^n$.*

According to Theorem O and Theorem MR, we have

THEOREM 2.4. *Let M be a connected semi-invariant submanifold of codimension 3 in a nonflat complex space form $M_{n+1}(c), c \neq 0$. If M satisfies $\mathfrak{L}_\xi \nabla = 0$ and that $\rho - c(n^2-1) - (t_r A)^2 - t_r A^2 \geq 0$, then M is locally congruent one of the following spaces : If $c > 0$, or if $c < 0$, then M is the same types as those in Theorem O and Theorem MR respectively, where $\mathfrak{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector ξ and ρ the scalar curvature of M.*

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