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ADDITIVE MAPPINGS ON OPERATOR ALGEBRAS PRESERVING SQUARE ABSOLUTE VALUES

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Abstract. Let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ denote the algebras of all bounded linear operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. We show that if $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ is an additive mapping satisfying $\phi(|A|^2) = |\phi(A)|^2$ for every $A \in \mathcal{B}(H)$, then there exists a mapping ψ defined by $\psi(A) = \phi(I)\phi(A)$, $\forall A \in \mathcal{B}(H)$ such that ψ is the sum of two *-homomorphisms one of which **C**-linear and the othere **C**-antilinear. We will also study some conditions implying the injective and rank-preserving of ψ.

1. Introduction

We are interested in characterizing properties linear operators on matrix algebras that leave certain invariant. This subject of linear preserver problems has been the focus of attention of many mathematicians [6]. And much research has been going on in this area. In recent years there has been also a considerable interest in linear preserver problems on algebras over infinite-dimensional spaces [1; 2; 3; 5; 8; 9].

Let $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces and let $\mathcal{B}(H)$ and $\mathcal{B}(K)$ denote the algebras of all bounded linear operators on $\mathcal{H}$ and $\mathcal{K}$, respectively. It is the aim of this paper to continue this work by studying additive mappings $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ that preserve square absolute value. We say that an additive mapping $\phi : \mathcal{B}(H) \longrightarrow$

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$\mathcal{B}(K)$ preserves square absolute value if the $\phi(|A|^2) = |\phi(A)|^2$ for every $A \in \mathcal{B}(H)$, where $\psi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ defined by $\psi(A) = \phi(I)\phi(A)$ is the sum of two *-homomorphisms one which **C**-linear and the other **C**-antilinear. We will also study some conditions implying the injectivity and rank-preserving of ψ.(Note that in [7, Theorem 1] it is assumed that $\mathcal{F}(H) \subset \text{ran}\ \Phi$ which we don't assume.) By a *-homomorphism we just mean a map $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ which preserves the ring structure and for which $\phi(A^*) = \phi(A)^*$ for every $A \in \mathcal{B}(H)$.

Let $\mathcal{B}_0(\mathcal{H})$ be the set of all compact operators on $\mathcal{H}$ and $N_3(\mathcal{H})$ be the set of all bounded linear nilpotent operators with nilindex 3, that is, if $N \in N_3(\mathcal{H})$ we have $N^3 = 0$ and for every $k > 0$, $N^k \neq 0$. We write $\sigma(T)$ for the spectrum of T.

2. The main results

In the following theorem we would like to characterize the ring homomorphisms ϕ which are additive mappings preserving square absolute value.

THEOREM 1. *Assume $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and let $\phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$ be an additive mapping preserving square absolute value. Then, there exists an (orthogonal) direct sum $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ such that $\mathcal{K}_0 = ker\phi(I)$ and ψ which $\psi(A) = \phi(I)\phi(A)$ satisfies the following results:*

1) $\psi(A) = 0 \oplus \psi_1(A) \oplus \psi_2(A)$ for all $A \in \mathcal{B}(H)$, where

*2) $\psi_1 : \mathcal{B}(H) \longrightarrow \mathcal{B}(\mathcal{K}_1)$ is a **C**-linear $*$-homomorphism.*

*3) $\psi_2 : \mathcal{B}(H) \longrightarrow \mathcal{B}(\mathcal{K}_2)$ is a **C**-antilinear *-homomorphism.*

*4) If $\phi(I)$ is an injective operator then, $\mathcal{K}_0 = \{0\}$ and only one of the summands $\mathcal{K}_1$ and $\mathcal{K}_2$ are nonzero. In this case ϕ is a **C**-linear or **C**-antilinear $*$-homomorphism. Moreover, $\psi(S) = \phi(S)$, for every self-adjoint operator $S \in \mathcal{B}(H)$.*

Proof. By step 1 of the proof of [7, Theorem 1], ϕ is an **R**-linear continuous map sending all positive and self-adjoint operators on $\mathcal{H}$ to the same types of operators on $\mathcal{K}$.

Now by a similar argument as in [10, Theorem 1] one can show that for all]commuting self-adjoint operators $S,\ T \in \mathcal{B}(H)$.

(1.1) $$\phi(iT)^*\phi(S) = -\phi(S)\phi(iT),$$

(1.2) $$\phi(S)\phi(I) = \phi(I)\phi(S).$$

Now suppose that S is a self-adjoint operator and t is an arbitrary element of $\mathbf{R}$, we have

$$e^{itS} = I + (iS)t - S^2/2\ t^2 + ...$$

$$\phi(e^{itS}) = \phi(I) + \phi(iS)t - \phi(S^2)/2\ t^2 + ..$$

$$\phi(e^{itS})^* = \phi(I) + \phi(iS)^*t - \phi(S^2)^*/2\ t^2 + ...$$

Since e^{itS} is unitary, by the assumption we get,

$$\begin{aligned}\phi(I) &= \phi(e^{itS})^*\phi(e^{itS}) \\ &= \phi(I) + [\phi(iS)^*\phi(I) + \phi(I)\phi(iS)]t \\ &- 1/2\ [\phi(I)\phi(S^2) + \phi(S^2)^*\phi(I) - 2\phi(iS)^*\phi(iS)]t^2 + ...\end{aligned}$$

This implies that

(1.3) $$\phi(iS)^*\phi(iS) = \phi(S^2)^*\phi(I) + \phi(I)\phi(S^2).$$

Define $\psi(A) = \phi(I)\phi(A),\ \forall A \in \mathcal{B}(H)$. According to (1.1) we obtain $\psi(iS)^* = -\psi(iS) = \psi(-iS) = \psi((iS)^*)$, That is, if $T^* = -T$, then $\psi(T)^* = \psi(T^*)$. Also by (1.2) $\psi(S) = \phi(I)\phi(S) = \phi(S)\phi(I) = (\phi(I)\phi(S))^* = \psi(S)^*$, for every self-adjoint operator S of $\mathcal{B}(H)$. Hence ψ is a *-preserving.

The relative (1.3) by using (1.2) implies that $\psi(S^2) = \phi(S^2)$, in particular for every positive operator and every self-adjoint operator S we have $\psi(S) = \phi(S)$. This implies the last part of the theorem.

$\phi(I)$ commutes with $\phi(A)$ for all $A \in \mathcal{B}(H)$, [10, Proof of theorem 1]. Therefore,

$$\begin{aligned}\psi(|A|^2) &= \phi(|A|^2)\phi(I) = |\phi(A)|^2\phi(I) = \phi(A)^*\phi(A)\phi(I) \\ &= \phi(A)^*\phi(I)^2\phi(A) = \psi(A)^*\psi(A) = |\psi(A)|^2,\end{aligned}$$

that is, ψ is an additive mapping preserving square absolute value. We show that ψ is a *-homomorphism. We note that ψ is a *-preserving. Since ψ is an additive mapping preserving square absolute value, $[\psi(iI)]^2 = -\psi(I^2) = -\psi(I)$ and also $\psi(iI)$ is a skew-self-adjoint operator and $\psi(I)^2 = \psi(I)$ and $\psi(I)$ is a self-addjoint operator. Let $\mathcal{K}_0 = ker\phi(I)$. Thus, we can get the following representations

$$\psi(I) = \begin{pmatrix} 0 & 0 \\ 0 & I_1 \end{pmatrix}, \ \psi(iI) = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix},$$

with respect to the direct sum $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_0^{\perp}$, where I_1 is an injective operator and $F^* = -F$. This implies that $\psi(iI)\mathcal{K} \subset \psi(I)\mathcal{K}$. Now, by [10, Theorem 2] ψ satisfies in statement 1-3 and the first part of 4. It remains that we prove the last part of 4.

Now, we suppose that ψ is a linear *-homomorphism. Hence

$$\phi(I)\phi(iA) = \psi(iA) = i\psi(A) = i\phi(I)\phi(A), \ \forall A \in \mathcal{B}(H).$$

It follows that $\phi(I)[\phi(iA) - i\phi(A)] = 0$. Since $Ker\phi(I) = 0$ we obtain $\phi(iA) = i\phi(A)$, i.e. ϕ is a linear map.

Let $A = U + iV \in \mathcal{B}(H)$. we have

$$\begin{aligned} \psi(A) &= \psi(U + iV) = \psi(U) + i\psi(V) \\ &= \phi(U) + i\phi(V) = \phi(U + iV) = \phi(A). \end{aligned}$$

This completes the proof of theorem.

If ψ is **C**-antilinear, then we can define the adjoint operator $\psi_0 : \mathcal{B}(K) \longrightarrow \mathcal{B}(K)$, given by $\psi_0(T) = T^*$. Note that ψ_0 is continuous and satisfies $\psi_0(\lambda T) = \bar{\lambda}\psi_0(T)$ for every $\lambda \in \mathbf{C}$. Set $\psi' = \psi_0 \circ \psi : \mathcal{B}(H) \longrightarrow \mathcal{B}(K)$. Then ψ' is **C**-linear, continuous if ψ is. Furthermore we have $\psi'(AB) = (\psi(AB))^* = \psi(B)^*\psi(A)^* = \psi'(B)\psi'(A)$ if ψ is homomorphism. Therefore ψ' is an antihomomorphism. In fact a **C**-linear *-ring homomorphism is an *-algebra homomorphism.

In the following theorem, we say a closed subspace $M \subset \mathcal{K}$ is a invariant subspace of the operator $T \in \mathcal{B}(K)$ if $TM \subset M$ and a reducing subspace if, in addition, $T(M^{\perp}) \subset M$. $\mathcal{B} \subset B(K)$ is reductive if the orthogonal complement of every closed invariant subspace of $\mathcal{B}$ is an invariant subspace of $\mathcal{B}$, and irreducible if no proper closed subspace is reducing for all $A \in \mathcal{B}$. $W^{(\aleph)}$ denotes the (orthogonal) direct sum of as many copies of an operator or space W as cardinality $\aleph$.

THEOREM 2. *Let $\mathcal{H}$, K, ϕ and ψ be as in Theorem 1. Fix $j = 1, 2$. Then there exist a direct sum $\mathcal{K}_j = \mathcal{K}_{j1} \oplus \mathcal{K}_{j2}$, a cardinality $\aleph_j$, and operator $U : \mathcal{K}_{ji} \to \mathcal{H}^{\aleph_j}$ such that*

1) $\mathcal{K}_{j1} \neq \{0\}$ if only if ψ_j is injective,

2) $\psi_j(A) = \psi_{j1}(A) \oplus \psi_{j2}(A)$,

3) $\psi_{ji}(A) = U^* A^{(\aleph_j)} U$, and U is unitary if $j = 1$ and antiunitary if $j = 2$,

4) ψ_{j2} is not injective.

Moreover, if $\phi(\mathcal{B}(H))$ is irredusible and contains at least one nonzero compact operator, then $\mathcal{K} = \mathcal{K}_j$ for some $j \in \{1, 2\}$ and ψ is of the form $A \to VAV^*$ for some unitary or antiunitary $V : \mathcal{H} \to K$.

Proof. Since $\psi(I)^2 = \psi(I)$, the proof of (1)-(4) follows by [10, Theorem 3].

To complete the proof of theorem, suppose that M be a reducing subspace of $\psi(\mathcal{B}(H))$. We show that M be a reducing subspace of $\phi(\mathcal{B}(H))$. Let $x \in M$ if $\phi(A)x \notin M$, for some $A \in \mathcal{B}(H)$, then $\phi(A)x \in M^{\perp}$. Hence $\phi(I)\phi(A)(x) \in M^{\perp}$, and this is impossible. Thus M is a invariant subspace of $\phi(\mathcal{B}(H))$, consequently reducing subspace of $\phi(\mathcal{B}(H))$. Therefore, $\psi(\mathcal{B}(H))$ is irreeducble, because $\phi(\mathcal{B}(H))$ is irreducible. It implies the proof of the last part of theorem by [10, Theorem 3].

THEOREM 3. *Let $\mathcal{H}$, K, ϕ and ψ be as in Theorem 1.]Furthermore, if $N_3(H) \subset Im\psi$ and $\phi|_{\mathcal{F}(H)}$ is not zero. Then each of mappings ψ_1 and ψ_2 are rank preserving map.*

Proof. It is sufficient that ψ_1 be verified, the other case can be treated in a similar fashion. Because ψ_1 is a **C**-linear ring homomorphism it is an algebra homomorphism. Since $ker\psi_1$ is a closed ideal in $\mathcal{B}(H)$ and $\mathcal{B}_0(\mathcal{H})$ is the minimal closed two sided ideal in $\mathcal{B}(H)$ [4, Theorem 5.5] we have $ker\Psi_1 = 0$ or $\mathcal{B}_0(\mathcal{H}) \subset ker\psi_1$. By hypothesis there exists a finite rank operator T such that $\psi(T)$ is not ze

Now, we show that ψ_1 maps rank one operators to rank one. If we choose $A \in \mathcal{B}(H)$ such that A has rank one we prove that $\psi_1(A)$ has rank one. It is sufficient to show that for every nilpotent operator N satisfying $N^3 = 0$, and every scalar $c \neq 1$, $\sigma(N + \psi_1(A)) \cap \sigma(N + c\psi_1(A)) = \{0\}$, [2, Corollary 1]. According to our assumption there is a operator $M \in \mathcal{N}_3(\mathcal{H})$ such that $\psi_1(M) = N$. Since $\psi_1(M^3) = \psi_1(M)^3 = N^3 = 0$. By the injectivitSince by theorem 3 ψ_1 is $\mathbf{C}$-linear contractive *-homomorphism we have , $\sigma(\psi_1(T)) \subset \sigma(T)$, for every $T \in \mathcal{B}(H)$, therefore,

$$\sigma(\psi_1(M) + \psi_1(A)) \cap \sigma(\psi_1(M) + c\psi_1(A)) = \{0\}.$$

Or,

$$\sigma(N + \psi_1(A)) \cap \sigma(N + c\psi_1(A)) = \{0\}.$$

So $\psi_1(A)$ is rank-one. Since ψ_1 is additive continuous map this completes the proof of the theorem.

REMARK. Note that in [7, Theorem 2] the author assumes the additive mapping $\Phi : \mathcal{B}(H) \longrightarrow \mathcal{B}(H)$ to be both bijective and $\phi(AB^*) = 0$ if and only if $\phi(A)\phi(B)^* = 0$. Note that from conditions (i) and (ii) of this theorem the injectivity of ϕ follows and therefore this assumptions is redundant. Let $\phi(A) = 0$, then $\phi(A)\phi(A)^* = 0$. Hence $AA^* = 0$. Therefore $||A||^2 = 0$. Consequently $A = 0$. This implies that ϕ is injective.

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