A Coupled Higher-Order Nonlinear Schrödinger Equation Including Higher-Order Bright and Dark Solitons

Jongbae Kim

We suggest a generalized Lax pair on a Hermitian symmetric space to generate a new coupled higher-order nonlinear Schrödinger equation of a dual type which contains both bright and dark soliton equations depending on parameters in the Lax pair. Through the generalized ways of reduction and the scaling transformation for the coupled higher-order nonlinear Schrödinger equation, two integrable types of higher-order dark soliton equations and their extensions to vector equations are newly derived in addition to the corresponding equations of the known higher-order bright solitons. Analytical discussion on a general scalar solution of the higher-order dark soliton equation is then made in detail.

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Jongbae Kim is with the Telecommunication Basic Research Laboratory, ETRI, Yusong P.O. Box 106, Taejon 305-600, Rep. of Korea.

(Phone: +82 42 860 5355, Email: jongbae@etri.re.kr)

Since the first observation of a solitary wave in a canal almost 170 years ago, the solitary wave phenomena have been reported in many fields of sciences. Especially the soliton in an optical fiber, which was first proposed in a nonlinear Schrödinger equation (NSE) in early 1970s [1], [2], has motivated many researchers to make use of it in the optical communication of the next generation. After a huge amount of study for the practical applications as well as for the academic interest, several soliton field experiments of 10 Gbps $\sim$ 40 Gbps communications have been carried out recently in Japan, USA, and Europe [3], respectively. For a higher rate transmission of pulses, the wavelength division multiplexing [4] could be also taken into account to conduct the soliton transmission experiment of 1 Tbps level in a laboratory [5].

In the ultrafast optical soliton system where a pulse is in general shorter than $T_0 \leq 100$ fs [4], higher-order effects such as the third-order dispersion [6], the self-steepening [7], and the self-frequency shift [8] need to be considered for the propagation of femtosecond pulses in a monomode optical fiber. Regarding the Hirota [9] and the Sasa-Satsuma [10] equations which are known to be the only two integrable types of the higher-order NSEs, the Painlevé integrability property [11], [12], an exact N-soliton solution [13], and solitary wave and shock solutions in the generalized phase function have been found [14]. Also for the description of the multimode transmission, it is necessary to accommodate degrees of freedom in cross-couplings between different modes of pulses in a vector NSE [15], [16]. Remarkably the vector solitons of polarization-locked states, which have been only of the theoretical interest [17], are observed experimentally [18]. In a combined point of view, the simultaneous inclusion of both the higher-order and the cross-coupling effects leads to a coupled higher-order NSE (CHONSE) which is found to be integrable only for special cases of coupling constants. The Hirota and the Sasa-Satsuma equations are extended to the limited vector forms of the CHONSEs as described in [19] and [11], [20], respectively. Using the concept of matrix potential [21], however, we have suggested a general extension of the Hirota and the Sasa-Satsuma equations and clarified their relationships [22] in association with a formalism of Hermitian symmetric spaces [23]. Applying the formalism, an infinite number of conserved quantities of the two integrable equations are obtained as well [24].

Besides the bright soliton mentioned above, the NSE also admits the existence of a dark soliton [25], [26] for the positive group velocity dispersion. Similar to the bright soliton, the dark pulse soliton possesses distinct properties [27] but the higher-order and/or the cross-coupling effects are not studied enough yet. In the present article, based on the Lax pair formalism on the Hermitian symmetric spaces, we will derive a CHONSE of a dual type that simultaneously contains higher-order dark soliton equations (HODSEs) as well as higher-order bright soliton equations. Generalized ways of consistent reduction will be applied to the CHONSE to find any of integrable types allowed by the formalism. Finally a general and theoretical solution will be discussed for the HODSE including the new types of the higher-order dark solitons.

In a monomode optical fiber, the propagation of the ultrashort pulse is governed by the NSE including the higher-order terms [28]

$$\begin{aligned} \overline{\partial}\psi &= i(\alpha_1\partial^2\psi + \alpha_2|\psi|^2\psi) \\ &+ \alpha_3\partial^3\psi + \alpha_4\partial(|\psi|^2\psi) + \alpha_5\psi\partial(|\psi|^2), \end{aligned} \quad (1)$$

where $\overline{\partial} \equiv \partial/\partial\bar{z}$ and $\partial \equiv \partial/\partial z$ are derivatives in retarded time coordinates ($\bar{z} = x, z = t - x/v$), and ψ is a slowly varying envelope function. The real coefficients α_i $(i = 1, 2, 3, 4)$ in (1) specify in sequence the effects of the group velocity dispersion, the self-phase modulation, the third order dispersion, and the self-steepening. The remaining coefficient α_5 in the last term is complex in general. The real and the imaginary parts of α_5 are due to the effect of the frequency-dependent radius of fiber mode and the effect of the self-frequency shift by stimulated Raman scattering, respectively. The higher-order NSE in (1) can be scaled by appropriate change of variables

$$z = c_1 z', \quad \bar{z} = c_2 \bar{z}', \quad \psi = c_3\psi' \quad (2)$$

with the real constants c_1, c_2, and c_3. The resulting normalized equation, omitting the notation of prime for convenience, is

$$\begin{aligned} \overline{\partial}\psi &= i(\pm\partial^2\psi + |\psi|^2\psi) \\ &+ \partial^3\psi + \beta_1\partial(|\psi|^2\psi) + \beta_2\psi\partial|\psi|^2 \end{aligned} \quad (3)$$

if the scaling constants are taken as

$$c_1 = \pm\frac{\alpha_3}{\alpha_1}, \quad c_2 = \pm\frac{\alpha_3^2}{\alpha_1^3}, \quad |c_3|^2 = \pm\frac{\alpha_1^3}{\alpha_2\alpha_3^2}. \quad (4)$$

Due to the presence of the couplings α_3, α_4, and α_5, the NSEs of the bright soliton and the dark soliton, which respectively correspond to $(+)$ and $(-)$ signs in the group velocity dispersion term in (3), are perturbatively corrected by the higher-order effects. The coefficients of the self-steepening and the self-frequency effect terms

$$\beta_1 = \pm\frac{\alpha_1\alpha_4}{\alpha_2\alpha_3}, \quad \beta_2 = \pm\frac{\alpha_1\alpha_5}{\alpha_2\alpha_3} \quad (5)$$

are the remaining free parameters to determine the integrability of (3). The same order of $(\pm)$ double signs are implied in (3)-(5). For the higher-order bright soliton equations, the two integrable types $\beta_1 = -\beta_2 = +3$ (Hirota case) [9] and $\beta_1 = -2\beta_2 = +3$ (Sasa-Satsuma case) [10] are well known already. However, since the integrability of the HODSE has not been studied yet, relevant issues will be discussed below.

For better understanding, we briefly review the definition of the Hermitian symmetric spaces [16], [23]. A symmetric space is a coset space G/K for Lie groups $G \supset K$. The associated Lie algebras $\mathbf{g}$, $\mathbf{k}$, and $\mathbf{m}$ of the coset space G/K are related by the decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$ and subject to the commutation relations

$$[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}, \quad [\mathbf{m}, \mathbf{m}] \subset \mathbf{k}, \quad [\mathbf{k}, \mathbf{m}] \subset \mathbf{m}. \quad (6)$$

A Hermitian symmetric space is the symmetric space G/K equipped with a complex structure. The Hermicity of G/K implies that there always exists an element T in the Cartan subalgebra of $\mathbf{g}$ whose adjoint action characterizes the properties of the subalgebras $\mathbf{k}$ and $\mathbf{m}$. That is, the adjoint action $J = \mathrm{ad}(T) \equiv [T, *]$ is a linear map which defines the subalgebra $\mathbf{k}$ as a kernel by $[T, \mathbf{k}] = 0$, and the subalgebra $\mathbf{m}$ by $[T, \mathbf{m}] \subset \mathbf{m}$ satisfying the complex structure condition, $J^2 = -I$, or $[T, [T, \mathbf{m}]] = -\mathbf{m}$ for any element of $\mathbf{m}$. In the present discussion we will restrict our concerns only to the Hermitian symmetric space $AIII$ of $G/K = SU(M+N)/(SU(M)\otimes SU(N)\otimes U(1))$ in general [22] so that the expression of a CHONSE becomes simply relevant to the extension of (3) to a vector equation of multicomponents.

Since the existence of a Lax pair admits the integrability of an equation that arises from the compatibility condition, we first propose a Lax pair on the Hermitian symmetric space

$$L_z = \partial + \epsilon_1 E + \lambda\epsilon_2 T \qquad (7)$$

$$L_{\bar z} = \overline{\partial} + U_0 + \lambda U_1 + \lambda^2 U_2 + \lambda^3 U_3 \qquad (8)$$

which generate a set of linear equations $L_z\Psi = L_{\bar z}\Psi = 0$. The entities in $L_{\bar z}$ are

$$\begin{aligned} U_0 &= \overline{\epsilon}_1(\partial\tilde{E} - \epsilon_1 E\tilde{E}) + \overline{\epsilon}_2(\partial^2 E - \epsilon_1[E, \partial E]) \\ &- 2\epsilon_1^2\overline{\epsilon}_2 E^3 \qquad (9) \\ U_1 &= -\epsilon_2\overline{\epsilon}_2(\partial\tilde{E} - \epsilon_1 E\tilde{E}) + \overline{\epsilon}_1\epsilon_2 E \qquad (10) \\ U_2 &= \epsilon_2^2(\frac{\overline{\epsilon}_1}{\epsilon_1}T - \overline{\epsilon}_2 E) \qquad (11) \\ U_3 &= -\frac{\epsilon_2^3\overline{\epsilon}_2}{\epsilon_1}T, \qquad (12) \end{aligned}$$

including also the element T in the Cartan subalgebra. Here E and $\tilde{E} \equiv [T, E]$ are extended field variables which belong to $\mathbf{m}$, while ϵ_1, ϵ_2, $\overline{\epsilon}_1$, and $\overline{\epsilon}_2$ are arbitrary complex constants and λ is the spectral parameter. The requirement of integrability, which is equivalent to the compatibility condition $[L_z, L_{\bar z}] = 0$ for all values of the parameter λ, leads to a generalized CHONSE

$$\begin{aligned} \overline{\partial}E &= \frac{\overline{\epsilon}_1}{\epsilon_1}\partial^2\tilde{E} - 2\epsilon_1\overline{\epsilon}_1 E^2\tilde{E} \\ &+ \frac{\overline{\epsilon}_2}{\epsilon_1}\partial^3 E - 3\epsilon_1\overline{\epsilon}_2(E^2\partial E + \partial E E^2) \qquad (13) \end{aligned}$$

in terms of the three constants ϵ_1, $\overline{\epsilon}_1$, and $\overline{\epsilon}_2$. To be noted is that the given Lax pair in (7)-(8) and the resulting CHONSE in (13), which are generalized in ϵ, $\overline{\epsilon}$ parameter space, accommodate the two types suggested in [22], [24] and therefore cover others in [11], [19], [20] in a consistent formulation, if specific choices are taken for the constants ϵ_1, $\overline{\epsilon}_1$, and $\overline{\epsilon}_2$. In the coset space $AIII$, the anti-Hermitian E and the diagonal T can be represented as

$$E = \begin{pmatrix} 0 & R & S \\ -R^\dagger & 0 & 0 \\ -S^\dagger & 0 & 0 \end{pmatrix}, \quad T = \frac{i}{2}\begin{pmatrix} I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{pmatrix} \qquad (14)$$

with complex matrices R, S of the same dimension. Then the CHONSE in (13) is decomposed into a set of two equations for R,

$$\begin{aligned} \overline{\partial}R &= i\left[\frac{\overline{\epsilon}_1}{\epsilon_1}\partial^2 R + 2\epsilon_1\overline{\epsilon}_1(RR^\dagger + SS^\dagger)R\right] + \frac{\overline{\epsilon}_2}{\epsilon_1}\partial^3 R \\ &+ 3\epsilon_1\overline{\epsilon}_2[(RR^\dagger + SS^\dagger)\partial R \\ &\quad + (\partial RR^\dagger + \partial SS^\dagger)R] \qquad (15) \end{aligned}$$

and S that is exactly symmetric under the interchange $R \leftrightarrow S$.

In regard of reduction [22], the procedures of reducing higher dimensional equations to lower dimensional equations in a consistent way are generalized in the present approach to find, if any, a new type of a CHONSE other than the known types. From (15) and its symmetric one for $R \leftrightarrow S$ one may guess that plausible candidates for the reduction of the equations may be $S = 0$, $S \sim R$, $S \sim R^*$, $S \sim R^T$, and $S \sim R^\dagger$ but rough estimations show that only the first three are allowed in view of the consistency after reduction. First of all a reduced equation for $S = 0$ is easily obtained directly from (15). Then let us suppose $S = e^{\phi(\bar z, z)}R$ with a general function $\phi(\bar z, z)$ for the second case. The algebraic calculation leads to another reduced equation

$$\begin{aligned} \overline{\partial}R &= i\left(\frac{\overline{\epsilon}_1}{\epsilon_1}\partial^2 R + 4\epsilon_1\overline{\epsilon}_1 RR^\dagger R\right) + \frac{\overline{\epsilon}_2}{\epsilon_1}\partial^3 R \\ &+ 6\epsilon_1\overline{\epsilon}_2\left(RR^\dagger\partial R + \partial RR^\dagger R\right) \qquad (16) \end{aligned}$$

with a condition $\partial\phi = \overline{\partial}\phi = 0$ for the two equations for R, S to be equivalent. Hence in the reduction of the type, only $\phi(\bar z, z) =$ constant is possible. For the third case if $S = e^{\phi(\bar z, z)}R^*$, the same requirement of well-defined equations for R, S, which are to be related by a complex conjugate each other, leads a new type of another reduced equation

$$\begin{aligned} \overline{\partial}R &= i\left(\frac{\overline{\epsilon}_1}{\epsilon_1}\partial^2 R + 2\epsilon_1\overline{\epsilon}_1 RR^\dagger R\right) + \frac{\overline{\epsilon}_2}{\epsilon_1}\partial^3 R \\ &+ 3\epsilon_1\overline{\epsilon}_2[(RR^\dagger + R^*R^T)\partial R \\ &\quad + (\partial RR^\dagger + \partial R^*R^T)R] \qquad (17) \end{aligned}$$

under a condition

$$\partial\phi = -\frac{2\overline{\epsilon}_1}{3\overline{\epsilon}_2}, \quad \overline{\partial}\phi = -\frac{4\overline{\epsilon}_1^3}{27\epsilon_1\overline{\epsilon}_2^2}. \qquad (18)$$

To be noted is that, the direct derivation of (17) is possible from reducing (15) because of the introduction of the local transformation as given in (18). Consequently the different ways of reduction from the extended CHONSE provide three kinds of the independent equations. In view of (1)-(5), needed quantities of the reduced equations can be calculated. That is, if $G/K = SU(2)/U(1)$, (i) $S = 0,\ R = \psi$ leads to $\alpha_1 = \frac{\bar{\epsilon}_1}{\epsilon_1},\ \alpha_2 = 2\epsilon_1\bar{\epsilon}_1,\ \alpha_3 = \frac{\bar{\epsilon}_2}{\epsilon_1},\ \alpha_4 = 6\epsilon_1\bar{\epsilon}_2,\ \alpha_5 = -6\epsilon_1\bar{\epsilon}_2,\ c_1 = \pm\frac{\bar{\epsilon}_2}{\bar{\epsilon}_1},\ c_2 = \pm\frac{\epsilon_1\bar{\epsilon}_2^2}{\bar{\epsilon}_1^3},\ |c_3|^2 = \pm\frac{1}{2}\cdot\frac{\bar{\epsilon}_1^2}{\epsilon_1^2\bar{\epsilon}_2^2}$, and $\beta_1 = -\beta_2 = \pm 3$, on the other hand if $G/K = SU(3)/(SU(2)\otimes U(1))$, (ii) $S = e^{\phi(\bar{z},z)}R,\ R = \psi$ leads to $\alpha_1 = \frac{\bar{\epsilon}_1}{\epsilon_1},\ \alpha_2 = 4\epsilon_1\bar{\epsilon}_1,\ \alpha_3 = \frac{\bar{\epsilon}_2}{\epsilon_1},\ \alpha_4 = 12\epsilon_1\bar{\epsilon}_2,\ \alpha_5 = -12\epsilon_1\bar{\epsilon}_2,\ c_1 = \pm\frac{\bar{\epsilon}_2}{\bar{\epsilon}_1},\ c_2 = \pm\frac{\epsilon_1\bar{\epsilon}_2^2}{\bar{\epsilon}_1^3},\ |c_3|^2 = \pm\frac{1}{4}\cdot\frac{\bar{\epsilon}_1^2}{\epsilon_1^2\bar{\epsilon}_2^2}$, and $\beta_1 = -\beta_2 = \pm 3$, and (iii) $S = e^{\phi(\bar{z},z)}R^*,\ R = \psi$ leads to $\alpha_1 = \frac{\bar{\epsilon}_1}{\epsilon_1},\ \alpha_2 = 2\epsilon_1\bar{\epsilon}_1,\ \alpha_3 = \frac{\bar{\epsilon}_2}{\epsilon_1},\ \alpha_4 = 6\epsilon_1\bar{\epsilon}_2,\ \alpha_5 = -3\epsilon_1\bar{\epsilon}_2,\ c_1 = \pm\frac{\bar{\epsilon}_2}{\bar{\epsilon}_1},\ c_2 = \pm\frac{\epsilon_1\bar{\epsilon}_2^2}{\bar{\epsilon}_1^3},\ |c_3|^2 = \pm\frac{1}{2}\cdot\frac{\bar{\epsilon}_1^2}{\epsilon_1^2\bar{\epsilon}_2^2}$, and $\beta_1 = -2\beta_2 = \pm 3$, respectively. In case of the bright soliton equations, real c_1, c_2, and c_3 always exist provided ϵ_1, $\bar{\epsilon}_1$, and $\bar{\epsilon}_2$ are real. In case of the dark soliton equations, the existence of real scaling requires especially

$$|c_3|^2 \sim -\frac{\bar{\epsilon}_1^2}{\epsilon_1^2\bar{\epsilon}_2^2} > 0. \tag{19}$$

The complex analysis in detail shows that c_1, c_2, and c_3 are real in the case of the dark solitons provided ϵ_1, $\bar{\epsilon}_1$, and $\bar{\epsilon}_2$ are purely imaginary. Conclusively speaking, if ϵ_1, $\bar{\epsilon}_1$, and $\bar{\epsilon}_2$ are real the CHONSE provides the higher-order bright soliton equations of the known Hirota and Sasa-Satsuma types but if the parameters are purely imaginary the same CHONSE provides the HODSEs of the two new types

$$\beta_1 = -\beta_2 = -3 \qquad \text{(type I)} \tag{20}$$

$$\beta_1 = -2\beta_2 = -3 \qquad \text{(type II)} \tag{21}$$

as well. Furthermore if $R = (\psi_1,\ \psi_2,\ \cdots,\ \psi_N)$ is simply substituted into (15) with $S = 0$ (or equivalently (16)) and into (17) and then normalized, the multicomponent representation leads to the N-coupled extension of (3),

$$\begin{aligned}
\bar{\partial}\psi_k &= i\left(\pm\partial^2\psi_k + \sum_{j=1}^{N}\psi_j^*\psi_j\psi_k\right) + \partial^3\psi_k \\
&+ \begin{cases} (\pm\frac{3}{2})\sum_{j=1}^{N}\psi_j^*\psi_j\partial\psi_k \\ (\pm 3)\sum_{j=1}^{N}\psi_j^*\psi_j\partial\psi_k \end{cases} \\
&+ \begin{cases} (\pm\frac{3}{2})\sum_{j=1}^{N}\psi_j^*\partial\psi_j\psi_k \\ (\pm\frac{3}{2})\sum_{j=1}^{N}\partial(\psi_j^*\psi_j)\psi_k \end{cases}
\end{aligned} \tag{22}$$

involving two vector HODSEs corresponding to the type I, II in (20)-(21) as well as the known types of the higher-order vector NSEs for the bright soliton [22]. No new type arises in a point of view of the normalization in (2)-(5) even though the reduction from a higher dimensionality of the coset space G/K is applied. It is also remarkable that the Lax pair in (7)-(8) implicitly accommodates dual types of the bright and the dark soliton equations, depending on whether the parameters ϵ_1, $\bar{\epsilon}_1$, and $\bar{\epsilon}_2$ are all real or all purely imaginary.

A transformation of variables from an unprimed to a double primed coordinate system and of the matrix E changes the form of the CHONSE in (13). Under the transformation

$$\begin{aligned}
& z \to z'' = z + pz, \quad \bar{z} \to \bar{z}'' = \bar{z}, \\
& E \to F = e^{\theta(\bar{z},z)T}Ee^{-\theta(\bar{z},z)T}
\end{aligned} \tag{23}$$

with a real constant p and a general function $\theta(\bar{z},z)$, the anti-Hermicity and the complex structure condition $[T,[T,F]] = -F$ are maintained for $E \to F \subset \mathbf{m}$ on the Hermitian symmetric space. In a consistent point of view, eliminating the unnecessary types of terms to form a new CHONSE for F after the transformation,

$$\begin{aligned}
\bar{\partial}F &= \frac{\bar{\epsilon}_1 - 3\bar{\epsilon}_2\partial\theta}{\epsilon_1}\partial^2\tilde{F} - 2\epsilon_1(\bar{\epsilon}_1 - 3\bar{\epsilon}_2\partial\theta)F^2\tilde{F} \\
&+ \frac{\bar{\epsilon}_2}{\epsilon_1}\partial^3 F - 3\epsilon_1\bar{\epsilon}_2(F^2\partial F + \partial F F^2)
\end{aligned} \tag{24}$$

imposes restrictions on p and $\theta(\bar{z},z)$,

$$p = 2\frac{\bar{\epsilon}_1}{\epsilon_1}(\partial\theta) - 3\frac{\bar{\epsilon}_2}{\epsilon_1}(\partial\theta)^2 \tag{25}$$

$$\partial^2\theta = 0 \tag{26}$$

$$\bar{\partial}\theta = -\frac{\bar{\epsilon}_1}{\epsilon_1}(\partial\theta)^2 + 2\frac{\bar{\epsilon}_2}{\epsilon_1}(\partial\theta)^3 \tag{27}$$

without any loss of generality. The notation of double prime is omitted in (24)-(27) for convenience. If (24) is compared with (13), in spite of the differences in the group velocity dispersion and the self-phase modulation terms generated by the point transformation, the two CHONSEs are nothing but equivalent to each other. Therefore all the results relevant for F including the analysis of reduction are obtained from the straightforward replacements $E \to F$, $R \to V$, $S \to W$, and $\bar{\epsilon}_1 \to \bar{\epsilon}_1 - 3\bar{\epsilon}_2\partial\theta$ in the corresponding equations, respectively. For example, regarding $|c_3|^2 \sim -\frac{(\bar{\epsilon}_1 - 3\bar{\epsilon}_2\partial\theta)^2}{\epsilon_1^2\bar{\epsilon}_2^2} > 0$ in analogy with (19), the same conclusion is implied for the characteristic of (24) depending on whether ϵ_1, $\bar{\epsilon}_1 - 3\bar{\epsilon}_2\partial\theta$, and $\bar{\epsilon}_2$ are real or purely imaginary.

It is crucial that the generalized Lax pair combined with the scaling transformation predicts the unprecedented types of the HODSEs and their vector equations for the purely imaginary constants. Since the Lax pair a priori assumes the existence of a solution, the corresponding equations can be solved analytically. The method of Bäcklund transformation which was used to solve the current formalism on the Hermitian symmetric spaces, as discussed in [22] for example, fails to apply to the HODSE in (3) because of the presence of the parameter $\epsilon_1 \neq 1$ in the Lax pair. In order for the HODSE to show the propagation of a soliton with a dark dip, however, one can imagine that the solution is necessary to have a globally similar form as the conventional one in the absence of the perturbative higher-order corrections. Therefore we suppose an ansatz in analog with the form described in [27], allowing for the most general number of parameters possible, as

$$\begin{aligned} \psi(\bar{z}, z) &= [A \tanh(a\bar{z} + bz + c) + iB] \\ &\times \exp(k\bar{z} - \omega z + \theta_0) \end{aligned} \tag{28}$$

with real constants $A,\ B,\ a,\ b,\ c,\ k,\ \omega$, and θ_0. This form is equivalent to a generalized dark soliton. Nontrivial relationships among the 6 dynamical parameters $A,\ B,\ a,\ b,\ k$, and ω will be determined if and only if the ansatz is indeed an analytical solution. If we put (28) into the HODSE in (3), the coefficient comparison of linearly independent functions requires constraints

$$\begin{aligned} a &= AB - 2\omega b + b(4b^2 - 3\omega^2) \\ &+ \beta_1(3bA^2 + bB^2 - \omega AB) + 2\beta_2 bA^2 \end{aligned} \tag{29}$$

$$b^2 = -\frac{1}{6}(3\beta_1 + 2\beta_2)A^2 \tag{30}$$

$$k = \omega^3 + \omega^2 - \beta_1(A^2 + B^2)\omega + A^2 + B^2 \tag{31}$$

$$\begin{aligned} 0 &= (6b^2 + \beta_1 A^2)\omega + 2(\beta_1 + \beta_2)bAB + 2b^2 \\ &- A^2 \end{aligned} \tag{32}$$

among the constants. The 4 relations in (29)-(32) seemingly implies that at least 2 dynamical parameters can remain independent in the solution. For the existence of a well-behaved solitary wave, the parameters should be properly defined also. First of all the condition

$$3\beta_1 + 2\beta_2 < 0 \tag{33}$$

is indispensable for the nontrivial $b^2 > 0$ in (30), and the real b is expressed as

$$b = \pm\sqrt{-\frac{(3\beta_1 + 2\beta_2)}{6}} \cdot A \tag{34}$$

in terms of A. If (34) is substituted into (32) then, ω can be solved as

$$\omega = \pm\sqrt{-\frac{(3\beta_1 + 2\beta_2)}{6}} \cdot B - \frac{3\beta_1 + 2\beta_2 + 3}{6(\beta_1 + \beta_2)} \tag{35}$$

in terms of B. Thus b and ω are determined appropriately, leaving restrictions on β_1 and β_2. In obtaining (35) another condition

$$\beta_1 + \beta_2 \neq 0, \text{ (excluding } \beta_1 = -\beta_2 = -3) \tag{36}$$

should be imposed for the existence of the well-defined ω. The intrinsic exception $\beta_1 = -\beta_2 = -3$ to the condition $\beta_1 + \beta_2 \neq 0$ is because ω can be arbitrary as undetermined in the case where $\beta_1+\beta_2 = 0$, $3\beta_1+2\beta_2+3 = 0$ are satisfied simultaneously. The remaining a and k are represented as they are if b and ω are replaced. A set of the seeming two solutions for an equation due to the presence of ($\pm$) signs in (34)-(35) are nothing but identical because they are related to each other by $\psi \leftrightarrow -\psi$ (i.e., $A \leftrightarrow -A$, $B \leftrightarrow -B$) in (28). Especially for the two types in (20)-(21) induced from the Lax pair, the relations among the constants can be more simplified as $a = (3\omega+1)AB - (\pm\frac{A}{\sqrt{2}})[A^2+3(\omega^2+B^2)+2\omega]$, $b = \pm\frac{A}{\sqrt{2}}$, $k = \omega^3 + \omega^2 + 3(A^2 + B^2)\omega + A^2 + B^2$, $\omega =$ arbitrary for the type I, and also $a = \pm A(-2A^2 - 3B^2 + \frac{1}{3})$, $b = \pm A$, $k = \pm B[4B^2 + (3A^2 - \frac{1}{3})] + \frac{2}{27}$, $\omega = \pm B - \frac{1}{3}$ for the type II, respectively. The same order of ($\pm$) double signs are implied from (34) through. Therefore the HODSE, which is in fact initiated by the dual implication of the Lax pair, accompanies the higher-order dark soliton as in (28) under the restrictions of coverage spanned by β_1 and β_2 in (33) and (36).

To summarize, based on the formulation of a Hermitian symmetric space we propose a generalized Lax pair and a resulting coupled higher-order nonlinear Schrödinger equation which accommodate others reported previously. The generalized ways of reduction and the normalizing transformation for the coupled higher-order nonlinear Schrödinger equation lead to two integrable types of dual equations in spite of its reducing dimensionality - the well-known Hirota and Sasa-Satsuma equations for the higher-order bright solitons and two new types of higher-order dark soliton equations, depending on whether parameters in the Lax pair are real or purely imaginary - and their extensions to N-component vector equations. The higher-order dark soliton equations and their vector extensions are unprecedented types derived from the Lax pair formalism. Finally we prove that a general scalar solution, covering the two new types, of the higher-order dark soliton equation exists theoretically under restricted conditions.

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Jongbae Kim received the B.S. and M.S. degrees in physics from Seoul National University at Seoul, Korea, in 1984 and 1986, respectively. In 1995, he received the Ph.D. degree in physics from the University of Maryland at College Park in USA. He then joined in ETRI in 1996 and is currently working on nonlinear optics and dispersion compensation in optical communication.