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THE SET OF ATTACHED PRIME IDEALS OF LOCAL COHOMOLOGY

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Abstract. In [2, 7.3.2], the set of attached prime ideals of local cohomology module $H^n_{\mathfrak{m}}(M)$ were calculated, where $(A, \mathfrak{m})$ be Noetherian local ring, M finite A-module and $\dim_A(M) = n$, and also in the special case in which furthermore A is a homomorphic image of a Gornestien local ring $(A', \mathfrak{m}')$ (see [2, 11.3.6]). In this paper, we shall obtain this set, by another way in this special case.

Throught this paper, A denotes a (non-trival) commutative Noetherian ring with identity. We use $\mathbb{N}$ (resp. $\mathbb{N}_0$) to denote the set of positive (resp. non-negative) integers.

DEFINITION 1. Let $M \neq 0$ be a finite A-module. Then $\operatorname{grade}_A(M) = \min\{i \in \mathbb{N}_0 : \operatorname{Ext}^i_A(M, A) \neq 0\}$ (the $\operatorname{grade}_A(M)$ of the zero submodule is infinity). (see [3, 1.2.5]). It follows from [3, 1.2.10(e)] that $\operatorname{grade}_A(M) = \operatorname{grade}(\operatorname{Ann}_A(M), A)$.

COROLLARY 2. *Let $(A, \mathfrak{m})$ be a cohen-Macaulay local ring, and $M \neq 0$ a finite A-module. Then* $\dim_A(M) = \dim A - \operatorname{grade}_A(M)$.

Proof. By [4, 17.4], we have

$$\begin{aligned}\dim A &= height(\operatorname{Ann}_A(M)) + \dim \frac{A}{\operatorname{Ann}_A(M)} \\ &= \operatorname{grade}_A(M) + \dim \frac{A}{\operatorname{Ann}_A(M)} \\ &= \operatorname{grade}_A(M) + \dim_A(M)\end{aligned}$$

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PROPOSITION 3. *(See [3, 1.2.4]). Let M, N be A-modules, and $\underline{x} = x_1, \ldots, x_n$ a weak M-sequence in* $\operatorname{Ann}_A(N)$. *Then*

$$\operatorname{Ext}^n_A(N, M) \simeq \operatorname{Hom}_A(N, \frac{M}{\underline{x}M}).$$

PROPOSITION 4. *(See [2, 11.2.6]) Let $(A, \mathfrak{m})$ be a Noetherian local ring which is a homomorphic image of a Gorenstein local ring $(A', \mathfrak{m}')$ of dimension n'. Let $f : A' \longrightarrow A$ be a surjective homomorphism. Set $E := E_A(\frac{A}{\mathfrak{m}})$ and $D :=$* $\operatorname{Hom}_A(-, E)$. *Then for each finite A-module M*

$$H^i_{\mathfrak{m}}(M) \simeq D(\operatorname{Ext}^{n'-i}_{A'}(M, A')) \text{ for all } i \in \mathbb{N}_0.$$

PROPOSITION 5. *Let $(A, \mathfrak{m})$ a local ring which is a homomorphic image of a Goerenstein local ring $(A', \mathfrak{m}')$ of dimension n', and M finite A-module. Then* $H^{\dim_A(M)}_{\mathfrak{m}}(M) \neq 0$.

Proof. Let $s' = \operatorname{grade}_{A'}(M)$. Then $H^{\dim_A(M)}_{\mathfrak{m}}(M)$ $= H^{\dim_{A'}(M)}_{\mathfrak{m}}(M) = H^{n'-s'}_{\mathfrak{m}}(M)$. Suppose that $H^{\dim_A(M)}_{\mathfrak{m}}(M) =$ 0. Then by proposition 4, $D(\operatorname{Ext}^{s'}_{A'}(M, A')) = 0$. It follows that $\operatorname{Ext}^{s'}_{A'}(M, A') = 0$, which is a contradiction. Therefore $H^{\dim_A(M)}_{\mathfrak{m}}(M) \neq 0$.

PROPOSITION 6.. *(See [2, Corollary 10.2.20]) Let $(A, \mathfrak{m})$ be a local ring (but not necessarily complete). Let M be a Noetherian A-module. Then* $\operatorname{Att}_A(D(M)) = \operatorname{Ass}_A(M)$, *where for any A-module N,* $\operatorname{Att}_A(N)$ *is the set of attached prime ideals of N.*

PROPOSITION 7. *(See [1, Ch. IV, Sec.1, Prop.10]. Let M be a finite A-module and N an A-module. Then*

$$\operatorname{Ass}_A(\operatorname{Hom}_A(M, N)) = \operatorname{Supp}_A(M) \cap \operatorname{Ass}_A(N).$$

Let $f : A \longrightarrow B$ be a surjective homomorphism of Noetherian rings and L be a finite A-module. Let for each $\mathfrak{p} \in$ $\operatorname{Spec}(A)$ *ideal $\mathfrak{p}B$ be the extension ideal $\mathfrak{p}$ to B. Then* $\operatorname{Ass}_B(M) = \{\mathfrak{p}B : \mathfrak{p} \in \operatorname{Ass}_A(L)\}$ *(see [4, Ex. 6.7]).*

The main purpose of this note is proved in the following theorem.

THEOREM 8. *Let $(A, \mathfrak{m})$ be a local ring which is a homomorphic image of a Gorenstien local ring $(A', \mathfrak{m}')$ of dimension n' and $M \neq 0$ a finite A-module. Let $f : A' \longrightarrow A$ be a surjective homomorphism. Then $H_{\mathfrak{m}}^{\dim_A(M)}(M) \neq 0$ and*

$$\begin{aligned}
&\mathrm{Att}_A(H_{\mathfrak{m}}^{\dim_A(M)}(M)) \\
&\quad = \left\{ \mathfrak{p}'A : \mathfrak{p}' \in \mathrm{Supp}_{A'}(M) \cap \mathrm{Ass}_{A'}(\frac{A'}{(x'_1, \ldots, x'_{s'})A'}) \right\} \\
&\quad = \left\{ \mathfrak{p} \in \mathrm{Ass}_A(M) : \dim \frac{A}{\mathfrak{p}} = \dim_A(M) \right\}
\end{aligned}$$

where $s' = \mathrm{grade}_{A'}(M)$ and $x'_1, \ldots, x'_{s'}$ is a maximal A'-sequence in $\mathrm{Ann}_{A'}(M)$ and $\mathrm{Att}_A(H_{\mathfrak{m}}^{\dim_A(M)}(M))$ is independent of the choice of maximal A'-sequence in $\mathrm{Ann}_{A'}(M)$.

Proof. By Definition 1 and Proposition 3, there is a maximal A'-sequence $x'_1, \ldots, x'_s$, in $\mathrm{Ann}_{A'}(M)$ such that $0 \neq \mathrm{Ext}_{A'}^{s'}(M, A') \simeq \mathrm{Hom}_{A'}(M, \frac{A'}{(x'_1, \ldots, x'_{x'})A'})$. By Proposition 4, we have $H_{\mathfrak{m}}^{n'-s'}(M) = D(\mathrm{Ext}_{A'}^{s'}(M, A'))$.

Now, from proposition 6 and proposition 7, we have

$$\begin{aligned}
&\mathrm{Att}_A(H_{\mathfrak{m}}^{\dim_A(M)}(M)) = \mathrm{Att}_A(H_{\mathfrak{m}}^{\dim_{A'}(M)}(M)) \\
&= \mathrm{Att}_A(H_{\mathfrak{m}}^{n'-s'}(M)) = \mathrm{Ass}_A(\mathrm{Ext}_{A'}^{s'}(M, A')) \\
&= \mathrm{Ass}_A \left(\mathrm{Hom}_{A'}(M, \frac{A'}{(x'_1, \ldots, x'_{s'})A'}) \right) \\
&= \left\{ \mathfrak{p}'A : \mathfrak{p}' \in \mathrm{Ass}_{A'} \left(\mathrm{Hom}_{A'}(M, \frac{A'}{(x'_1, \ldots, x'_{s'})A'}) \right) \right\} \\
&= \left\{ \mathfrak{p}'A : \mathfrak{p}' \in \mathrm{Supp}_{A'}(M) \cap \mathrm{Ass}_{A'} \left(\frac{A'}{(x'_1, \ldots, x'_{s'})A'} \right) \right\}.
\end{aligned}$$

Now, in view of [4, 17.3], $\mathfrak{p}' \in \mathrm{Supp}_{A'}(M) \cap \mathrm{Ass}_{A'} \left(\frac{A'}{(x'_1, \ldots, x'_{s'}A'} \right)$ if only if $\mathrm{grade}(\mathfrak{p}', A') = s'$ and $\mathrm{Ann}_{A'}(M) \subseteq \mathfrak{p}'$, and by [4, 17.4], if only if $\mathrm{ht}(\mathfrak{p}') = \mathrm{ht}(\mathrm{Ann}_{A'}(M)) = s'$ and $\mathrm{Ann}_{A'}(M) \subseteq \mathfrak{p}$, and so by [4, 17.4] if only if $\dim(A') - \dim(\frac{A'}{\mathfrak{p}'}) = \dim A' - \dim \left(\frac{A'}{\mathrm{Ann}_{A'}(M)} \right)$

and $\mathfrak{p}'$ is minimal element of $\mathrm{Supp}_{A'}(M)$, and if only if $\dim(\frac{A'}{\mathfrak{p}'}) = \dim(\frac{A'}{\mathrm{Ann}_{A'}(M)})$ and $\mathfrak{p}'$ is minimal element of $\mathrm{Ass}_{A'}(M)$.

Since $\dim(\frac{A'}{\mathfrak{p}'}) = \dim(\frac{A}{\mathfrak{p}'A})$, $\dim_{A'}(M) = \dim_A(M)$ and $\mathfrak{p}' \in \mathrm{Ass}_{A'}(M)$ if only if $\mathfrak{p}'A \in \mathrm{Ass}_A(M)$, hence

$$\begin{aligned}
&\mathrm{Att}_A(H_{\mathfrak{m}}^{\dim_A(M)}(M)) \\
&= \left\{ \mathfrak{p}A' : \mathfrak{p}' \in \mathrm{Supp}_{A'}(M) \cap \mathrm{Ass}_{A'} \left(\frac{A'}{(x'_1, \ldots, x'_{s'})A'} \right) \right\} \\
&= \left\{ \mathfrak{q} \in \mathrm{Ass}_A(M) : \dim \frac{A}{\mathfrak{q}} = \dim_A(M) \right\}.
\end{aligned}$$

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