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ASYMPTOTIC BEHAVIOUR OF IDEALS RELATIVE TO SOME MODULES OVER A COMMUTATIVE NOETHERIAN RING

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Abstract. Let E be an injective module over a commutative Noetherian ring A. In this paper we will show that if I is a regular ideal, then the sequence of sets

$$Ass_A((I^n)^{\star(E)}/I^n), \quad n \in N$$

is ultimately constant. Also we obtain some related results. (Here for an ideal J of A, $J^{\star(E)}$ denotes the integral closure of J relative to E.

1. Introduction

Throughout this paper A denotes a commutative Noetherian ring (with a non-zero identity) and E an injective A-module.

The important ideas of reduction and integral closure for ideals in a commutative Noetherian ring were introduced by Northcott and Rees in [10]; a brief and direct approach to their theory is given in [12, (1.1)] and it is appropriate to summurize some of those facts.

Let I be an ideal of A. Then I is a reduction of the ideal J of A if $I \subseteq J$ and there exists an integer $s \in N$ (we use the notation N to denote the set of positive integers) such that $IJ^s = J^{s+1}$. An element x of A is said to be integrally dependent on I if there exists $s \in N$ and elements $c_1, ..., c_s \in A$ with $c_i \in I^i$ for $i = 1, ..., s$ such that

$$x^s + c_1x^{s-1} + c_{s-1}x + c_s = 0.$$

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The set of all elements of A which are integrally dependent on I is an ideal of A, called the integral closure of I, and denoted by $(I)^{-}$. In fact it is the largest ideal of A which has I as a reduction.

Let I be an ideal of A and H an injective or an Artinian A-module. Then I is said to be a reduction of the ideal J of A relative to H if $I \subseteq J$ and there exists a positive integer s such that

$$(0 :_H IJ^s) = (0 :_H J^{s+1}).$$

An element x of R is said to be integrally dependent on I relative to H if there exists a positive integer s such that

$$(0 :_H \sum_{i=1}^{s} x^{s-i} I^i) \subseteq (0 :_H x^s).$$

The set of elements of A which are integrally dependent on I relative to H is an ideal of A, called the integral closure of I relative to H, and denoted by $I^{\star(H)}$. It is the largest ideal of A which has I as a reduction relative to H. (See [3] and [13].)

We shall follow Macdonald's terminology (see [6]) concerning secondary representation. So whenever an A-module L has a secondary representation, then the set of attached primes of L, which is uniquely determined, is denoted by $Att_A(L)$.

In [4] H. Ansari-Toroghy showed that if I is an ideal of A, then the sequences of sets

$$Ass_A(A/(I^n)^{\star(E)}), \quad n \in N,$$

is ultimately constant. Let us denote the ultimate constant value of this sequence by $\bar{As}^{\star}(I, E)$.

Let I be a regular ideal of A. In this paper we will show that the sequence of sets

$$Ass_A((I^n)^{\star(E)}/I^n), \quad n \in N,$$

is ultimately constant. Let us denote the ultimate constant value of this sequence by $Cs^{\star}(I, E)$. Then we show that

$$As^{\star}(I, A) = \bar{As}^{\star}(I, E) \cup Cs^{\star}(I, E).$$

(Here $As^\star(I,A)$ denotes the ultimate constant value of the sequence $Ass_A(A/I^n)$, $n \in N$.) This is simillar to [7, 11.19]. Also we will show that for each $n \in N$,

$$Ass_A(A/(I^n)^{\star(E)} = Ass_A((I^{n-1})^{\star(E)}/(I^n)^{\star(E)}.$$

Furthermore, we get some similar results related to Noetherian A-modules.

Throughout the remainder of this paper I will denote an ideal of A.

2. Previous results

We recall from [5], [11], and [7] that the sequences of sets

$$Ass_A(A/I^n), \quad Ass_A(A/\bar{I^n}), \quad Ass_A(\bar{I^n}/I^n), \quad \in N$$

are ultimately constant. We will denote their ultimate constant values by

$$As^\star(I,A), \quad \bar{As}^\star(I,A), \text{ and } Cs^\star(I,A).$$

DEFINITION 2.1. (See [14, (1.2)].) Let L be a Noetherian module over a commutative ring R (with identity). We say that I is a reduction of the ideal J of R relative L if $I \subseteq J$ and there exists $s \in N$ such that $IJ^sL = J^{s+1}L$. An element x of R is said to be integrally dependent on I relative to L if there exists $s \in N$ such that

$$x^sL \subseteq \sum_{i=1}^{s} x^{s-i}I^iL.$$

The set of elements of A which are integrally dependent on I relative to L is an ideal of R, called the integral closure of I relative to L, and denoted by $I^{-(L)}$. It is the largest ideal of R which has I as a reduction relative to L.

REMARK 2.2.

(i) Let M be a finitely generated A module. Then $Hom_A(M, E)$ has a secondary representation and we have

$$Att_A(Hom_A(M, E) \\ = \{P \in Ass_A(M) : P \subseteq Q \text{ for some } Q \in Ass_A(E)\}.$$

(See [1, Theorem 2.1] and [9, Lemma 1].)

(ii) The sequence of sets

$$Ass_A(A/(I^n)^{\star(E)}), \quad n \in N$$

is increasing and ultimately constant. We will denote it's ultimate constant value by $\bar{As}^\star(I, E)$. Further we have

$$\bar{As}^\star(I, E) = \{P \in \bar{As}^\star(I, A) : P \subseteq Q \text{ for some } Q \in Ass_A(E)\}.$$

(See [4, Theorem 3.1].)

(iii) For any $P \in Ass_A(E)$ we have

$$I^{\star(E)}A_P = \bar{I}A_P.$$

(See [3, (2.5) and (2.6)].)

(iv) Let $\phi : R \to S$ be a ring homomorphisim, where R and S are respectively a commutative and a commmutative Noetherian ring. Let $\phi^\star : Spec(S) \to Spec(R)$ be the induced map, and let H be an S-module. Then we have

$$Ass_R(H) = \phi^\star(Ass_S(H)).$$

(See [8, (3.2)].)

3. Consequences of the results of McAdam

THEOREM 3.1. *(We recall that E is an injective A-module.) Further let M be a finitely generated A-module.*

(i) Let

$$A(n) = Ass_A(A/(I^n)^{\star(E)} \text{ and } B(n) = Ass_A((I^{n-1})^{\star(E)}/(I^n)^{\star(E)}.$$

Then for each $n \in N$, $A(n) = B(n)$.

(ii) The sequence of sets

$$Ass_A(A/(I^n)^{-(M)}), \quad n \in N$$

is increasing and ultimately constant.

(iii) Let

$$A(n) = Ass_A(A/(I^n)^{-(M)}) \text{ and}$$
$$B(n) = Ass_A((I^{n-1})^{-(M)}/(I^n)^{-(M)}).$$

Then for each $n \in N$ *we have* $A(n) = B(n)$.

Proof. (i) It is clear that for each $n \in N$ we have $B(n) \subseteq A(n)$. To see the reverse inclusion, let $P \in A(n)$. Then by Remark 2. 2 (ii), $P \in \bar{As}^{\star}(I, A)$ and $P \subseteq Q$ for some $Q \in Ass_A(E)$. Now by using 2. 2 (iii), we have

$$P \in Ass_A(A/(I^n)^{\star(E)}) \text{ if only if } PA_Q \in Ass_{A_Q}(A_Q/(I^n)^{-}A_Q).$$

It implies that $P \in Ass_A(A/(I^n)^{-})$. It turns out that $P \in Ass_A((I^{n-1})^{-}/(I^n)^{-})$ by [7, 11. 28]. Hence we have

$$PA_Q \in Ass_{A_Q}((I^{n-1})^{-}A_Q/(I^n)^{-}A_Q)$$
$$= Ass_{A_Q}((I^{n-1})^{\star(E)}A_Q/(I^n)^{\star(E)}A_Q).$$

It implies that $P \in Ass_A((I^{n-1})^{\star(E)}/(I^n)^{\star(E)}) = B(n)$.

(ii) Let $\hat{A} = A/(0 :_A M)$ and $\hat{I} = (I + (0_A M))/(0 :_A M)$. It is clear that $(I)^{-(M)} \supseteq (0 :_A M)$. Now

$$(I)^{-(M)}/(0 :_A (M)) = (\hat{I})^{-}$$

by [14, (1. 6)]. Here $(\hat{I})^{-}$ denotes the integral closure of $\hat{I}$ in $\hat{A}$. Hence we have

$$A/(I^n)^{-(M)} \cong \hat{A}/((\hat{I})^n)^{-}.$$

But the sequence of sets

$$Ass_{\hat{A}}(\hat{A}/((\hat{I})^n)^-), \quad n \in N$$

is increasing and ultimately constant by [11]. It implies that the sequence of sets

$$Ass_A(A/(I^n)^{-(M)}) = Ass_A(\hat{A}/((\hat{I})^n)^-), \quad n \in N$$

is increasing and ultimately constant by Remark (2. 2) (iv).

(iii) It is clear that for each $n \in N$, $B(n) \subseteq A(n)$. Now let $P \in A(n)$ and $\hat{A}$ and $\hat{I}$ be as in (ii). Then by our arguments in (ii), $P \in Ass_A(\hat{A}/((\hat{I})^n)^-)$. So we have $P = Q \cap A$ for some $Q \in Ass_{\hat{A}}(\hat{A}/((\hat{I})^n)^-)$ by Remark 2. 2 (iv). It implies that $Q \in Ass_{\hat{A}}(((\hat{I})^{n-1})^-/((\hat{I})^n)^-)$ by [7, (11. 28)]. Hence

$$Q \cap A = P \in Ass_A(((\hat{I})^{n-1})^-/((\hat{I})^n)^-)$$

by 2. 2 (iv). But

$$(I^{n-1})^{-(M)}/(I^n)^{-(M)} \cong ((\hat{I})^{n-1})^-/((\hat{I})^n.$$

So we have $P \in B(n)$ and the proof is complete.

4. The results

THEORM 4.1. *Suppose that I contains a non-zerodivisor on A and H is an injective or an Artinian A-module. Then the sequence of sets*

$$Ass_A((I^n)^{\star(H)}/I^n), \quad n \in N,$$

is ultimately constant. We will denote it's ultimate constant value by $Cs^\star(I, H)$.

Proof. We apply the same technique used in [7, 11. 16]. Now there exists a positive integer k such that for $n \geq k$, $(I^{n+1} :_A I) = I^n$. Let $n \geq k$ and $P \in Ass_A(I^n)^{\star(H)}/I^n)$. Then thre exists an element $c \in (I^n)^{\star(H)}$ such that $P = (I^n :_A c)$. It implies that $P = (I^{n+1} :_A cI)$. Now we have

$$cI \subseteq (I^n)^{\star(H)} I \subseteq (I^{n+1})^{\star(H)}.$$

To see this let $xy \in (I^n)^{\star(H)} I$, where $x \in (I^n)^{\star(H)}$ and $y \in I$. Then there exists a positive integer s such that

$$(0 :_H \sum_{i=1}^{s} x^{s-i} (I^n)^i) \subseteq (0 :_H x^s).$$

So we get

$$\begin{aligned}(0 :_H \sum_{i=1}^{s} (xy)^{s-i} y^i (I^n)^i) &= ((0 :_H \sum_{i=1}^{s} x^{s-i} (I^n)^i) :_H y^s) \\ &\subseteq (0 :_H (xy)^s).\end{aligned}$$

Since $y^i (I^n)^i \subseteq (I^{n+1})^i$,

$$\begin{aligned}(0 :_H \sum_{i=1}^{s} (xy)^{s-i} (I^{n+1})^i) &\subseteq (0 :_H \sum_{i=1}^{s} (xy)^{s-i} y^i (I^n)^i)) \\ &\subseteq (0 :_H (xy)^s).\end{aligned}$$

Hence we have $xy \in (I^{n+1})^{\star(H)}$. It implies that $P \in Ass_A((I^{n+1})^{\star(H)}/I^{n+1})$. Hence for $n \geq k$,

$$Ass_A((I^n)^{\star(H)}/I^n)$$

becoms an increasing sequence. Now the result follows from the fact that

$$Ass_A((I^{n+1})^{\star(H)}/I^n) \subseteq Ass_A(A/I^n) \subseteq As^{\star}(I, A).$$

So the proof is complete.

THEOREM 4.2. *Suppose that I contains a non-zerodivisor on A. Then we have*

$$As^{\star}(I,A) = \bar{As}^{\star}(I,E) \cup Cs^{\star}(I,E).$$

Proof. It is clear that the right hand side is contained in the left because $\bar{As}^{\star}(I,E) \subseteq As^{\star}(I,A)$ by Remark 2.2 (ii) and that

$$Ass_A((I^n)^{\star(E)}/I^n) \subseteq Ass_A(A/I^n).$$

Now from the exact sequence

$$0 \to (I^n)^{\star(E)}/I^n \to A/I^n \to A/(I^n)^{\star(E)} \to 0,$$

we get

$$Ass_A(A/I^n) \subseteq Ass_A((I^n)^{\star(E)}/I^n) \cup Ass_A(A/(I^n)^{\star(E)}).$$

But as we have seen, the above three sequences are ultimately constant. Hence by the above arguments we have

$$As^{\star}(I,A) \subseteq \bar{As}^{\star}(I,E) \cup Cs^{\star}(I,E)$$

as desired. This completes the proof.

CROLLARY 4.3. *Suppose I contains a non-zerodivisor and let* $\bar{As}^{\star}(I,A) \cap Cs^{\star}(I,A) = \emptyset$. *Then we have*

$$Cs^{\star}(I,A) \subseteq Cs^{\star}(I,E).$$

Proof. By Remark 2.2 (iii),

$$\bar{As}^{\star}(I,E) \subseteq \bar{As}^{\star}(I,A).$$

So the result follows from 4.2 by using the fact that

$$As^{\star}(I,A) = Cs^{\star}(I,A) \cup \bar{As}^{\star}(I,A).$$

COROLLARY 4.4. *(see [2, Theorem 3.2]). Suppose that I contains a non-zerodivisor on A. Then the sequence of sets*

$$Att_A(0 :_E I^n)/(0 :_E (I^n)^{\star(E)}), \quad n \in N$$

is ultimately constant. If we denote the ultimate constant value the above sequence by $Ct^\star(I, E)$, then we have

$$Ct^\star(I, E) = \{P \in Cs^\star(I, E) : P \subseteq Q \text{ for some } Q \in Ass_A(E)\}.$$

Proof. We have

$$(0 :_E I^n)/(0 :_E (I^n)^{\star(E)}) \cong Hom_A((I^n)^{\star(E)}/I^n, E).$$

By Remark 2.2 (i),

$$Att_A((0 :_E I^n)/(0 :_E (I^n)^{\star(E)})) =$$

$$\{P \in Ass_A((I^n)^{\star(E)}/I^n) : P \subseteq Q \text{ for some } Q \in Ass_A(E)\}.$$

Now the result follows from Theorem 4.1 and 4.2.

THEOREM 4.5. *Let M be a finitely generated A-module. Further suppose that I contains a non-zerodivisor on A. Then the sequence of sets*

$$Ass_A((I^n)^{-(M)}/I^n), \quad n \in N,$$

is ultimately constant.

Proof. We can use similar arguments as in the proof of 3.1. Hence we omit the proof.

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