Honam Mathematical J. **23**(2001), No. 1,pp. 113–132

A REMARK ON G–INVARIANT MINIMAL HYPERSURFACES WITH CONSTANT SCALAR CURVATURES IN S^5

JAE-UP SO

Dept. of Mathematics,
Chonbuk National University, Chonju, 561-756, Korea.
E-mail : jaeup @ moak.chonbuk.ac.kr.

Abstract. We prove an equality that holds on G-invariant minimal hypersurfaces with constant scalar curvatures in S^5.

0. Introduction

Let M^n be a closed minimally immersed hypersurface in the unit sphere S^{n+1}, and h its second fundamental form. Denote by R and S its scalar curvature and the square norm of h, respectively. It is well known that $S = n(n-1) - R$ from the structure equations of both M^n and S^{n+1}. In particular, S is constant if and only if M has constant scalar curvature. In 1968, J. Simons [8] observed that if $S \leq n$ everywhere and S is constant, then $S \in \{0, n\}$. Clearly, M^n is an equatorial sphere if $S = 0$. And when $S = n$, M^n is indeed a product of spheres, due to the works of Chern, do Carmo, and Kobayashi [3] and Lawson [5].

We are concerned about the following conjecture posed by Chern [9].

Received May 7, 2001.
1991 AMS Subject Classification : 53C42.
Key words and phrases : scalar curvature, G-invariant minimal hypersurface, square norm.
This paper was supported by research funds of Chonbuk National University in 1999.

CHERN CONJECTURE. *For any $n \geq 3$, the set R_n of the real numbers each of which can be realized as the constant scalar curvature of a closed minimally immersed hypersurface in S^{n+1} is discrete.*

C. K. Peng and C. L. Terng [7] proved

THEOREM [Peng and Terng, 1983]. *Let M^4 be a closed minimally immersed hypersurface with constant scalar curvature in S^5. If $S > 4$, then $S > 4 + \frac{1}{48}$.*

S. Chang [2] proved

THEOREM [Chang, 1993]. *A closed minimally immersed hypersurface with constant scalar curvature in S^4 is either an equatorial 3-sphere, a product of spheres, or a Cartan's minimal hypersurface. In particular, $R_n = \{0, 3, 6\}$.*

Let $G \simeq O(k) \times O(p) \times O(q) \subset O(k+p+q)$ and set $k+p+q = n+2$. Then W. Y. Hsiang [4] investigated G-*invariant, minimal hypersurfaces, M^n* in S^{n+1}, by studying their generating curves, M^n/G, in the orbit space S^{n+1}/G and proved

THEOREM [Hsiang, 1987]. *For each dimension $n \geq 3$, there exist infinitely many, mutually noncongruent closed G-invariant minimal hypersurfaces in S^{n+1}, where $G \simeq O(k) \times O(k) \times O(q)$ and $k = 2$ or 3.*

We are going to study G-invariant minimal hypersurfaces, in stead of minimal ones, with constant scalar curvatures in S^5. Let M^4 be a G-invariant hypersurface in S^5. Then, we proved in [6] that there is a local orthonormal frame field $\{e_A\}$ in S^5 such that after restriction to M^4, the $e_1, \dots, e_4$ are tangent to M^4 and $h_{ij} = 0$ if $i \neq j$.

In this paper, we shall prove the following equality that holds under such a frame field and plays an important role in studying G-invariant minimal hypersurfaces with constant scalar curvatures in S^5:

THEOREM. *Let M^4 be a G-invariant minimal hypersurface with constant scalar curvature in S^5. Then, there exists a local or-*

thonormal frame field under which everywhere the following equality holds

$$\sum_{i,j,l,m} h_{ijlm}^2 = S(S-4)(S-11) + 3(A-2B),$$

where

$$A = \sum_{i,j,k} h_{ijk}^2 h_{ii}^2 \quad \text{and} \quad B = \sum_{i,j,k} h_{ijk}^2 h_{ii} h_{jj}.$$

REMARK. The above equality also appeared in C. K. Peng and C. L. Terng [7] holds at some one point. They proved it by using the fact $\Delta h_{ij} = \sum_k h_{ijkk}$ and $\Delta h_{ijl} = \sum_m h_{ijlmm}$, i.e., by using a frame field $\{v_A\}$ such that $\nabla_{v_i} v_j = 0$ for all i, j. But for our frame field $\{e_A\}$, in general $\Delta h_{ij} \neq \sum_l h_{ijll}$ and $\Delta h_{ijl} \neq \sum_m h_{ijlmm}$. Moreover, the above equality holds everywhere.

1. Preliminaries.

Let M^n be a manifold of dimension n immersed in a Riemannian manifold N^{n+1} of dimension $n+1$. Let $\overline{\nabla}$ and $\langle\, ,\, \rangle$ be the connection and metric tensor respectively of N^{n+1} and let $\bar{\mathcal{R}}$ be the curvature tensor with respect to the connection $\overline{\nabla}$ on N^{n+1}. Choose a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in N^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n. Denote the dual coframe by $\{\omega_A\}$. Here we will always use $i, j, k, \ldots,$ for indices running over $\{1, 2, \ldots, n\}$ and $A, B, C, \ldots,$ over $\{1, 2, \ldots, n+1\}$.

As usual, the *second fundamental form* h and the *mean curvature* H of M^n in N^{n+1} are respectively defined by

$$h(v, w) = \langle \overline{\nabla}_v w,\, e_{n+1} \rangle \quad \text{and} \quad H = \sum_i h(e_i,\, e_i).$$

And the *scalar curvature* $\bar{R}$ of N^{n+1} is defined by

$$\bar{R} = \sum_{A,B} \langle \bar{\mathcal{R}}(e_A,\, e_B) e_B,\, e_A \rangle.$$

Then the structure equations of N^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} K_{ABCD}\, \omega_C \wedge \omega_D,$$

where $K_{ABCD} = \langle \bar{\mathcal{R}}(e_A, e_B)e_D, e_C \rangle$. When N^{n+1} is the unit sphere S^{n+1}, we have

$$K_{ABCD} = \delta_{AC}\, \delta_{BD} - \delta_{AD}\, \delta_{BC}.$$

Next, we restrict all tensors to M^n. First of all, $\omega_{n+1} = 0$ on M^n. Then

$$\sum_i \omega_{(n+1)i} \wedge \omega_i = d\omega_{n+1} = 0.$$

By Cartan's lemma, we can write

$$\omega_{(n+1)i} = -\sum_j h_{ij}\, \omega_j.$$

Here,

$$\begin{aligned} h_{ij} &= -\omega_{(n+1)i}(e_j) = -\langle \overline{\nabla}_{e_j} e_{n+1}, e_i \rangle \\ &= \langle \overline{\nabla}_{e_j} e_i, e_{n+1} \rangle = h(e_j, e_i) = h(e_i, e_j). \end{aligned}$$

Second, from

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_l \omega_{il} \wedge \omega_{lj} - \frac{1}{2} \sum_{l,m} R_{ijlm}\, \omega_l \wedge \omega_m,$$

we find the curvature tensor of M^n is

$$R_{ijlm} = K_{ijlm} + h_{il}\, h_{jm} - h_{im}\, h_{jl}. \tag{1.1}$$

If M^n is a piece of minimally immersed hypersurface in the unit sphere S^{n+1} and R is the scalar curvature of M^n, then we have

$$R = n(n-1) - S, \tag{1.2}$$

where $S = \sum_{i,j} h_{ij}^2$ is the *square norm* of h.
Given a symmetric 2-tensor $T = \sum_{i,j} T_{ij}\, \omega_i\, \omega_j$ on M^n, we also define its covariant derivatives, denoted by ∇T, $\nabla^2 T$ and $\nabla^3 T$, etc. with components $T_{ij,k}$, $T_{ij,kl}$ and $T_{ij,klp}$, respectively, as follows:
(1.3)

$$\sum_k T_{ij,k}\,\omega_k = d\,T_{ij} + \sum_s T_{sj}\,\omega_{si} + \sum_s T_{is}\,\omega_{sj},$$

$$\sum_l T_{ij,kl}\,\omega_l = d\,T_{ij,k} + \sum_s T_{sj,k}\,\omega_{si} + \sum_s T_{is,k}\,\omega_{sj} + \sum_s T_{ij,s}\,\omega_{sk},$$

$$\begin{aligned}\sum_p T_{ij,klp}\,\omega_p = d\,T_{ij,kl} &+ \sum_s T_{sj,kl}\,\omega_{si} + \sum_s T_{is,kl}\,\omega_{sj}\\ &+ \sum_s T_{ij,sl}\,\omega_{sk} + \sum_s T_{ij,ks}\,\omega_{sl}.\end{aligned}$$

In general, the resulting tensors are no longer symmetric, and the rule to switch sub-index obeys the Ricci formula as follows:
(1.4)

$$T_{ij,kl} - T_{ij,lk} = \sum_s T_{sj}\,R_{sikl} + \sum_s T_{is}\,R_{sjkl},$$

$$\begin{aligned}T_{ij,klp} - T_{ij,kpl} &= \sum_s T_{sj,k}\,R_{silp} + \sum_s T_{is,k}\,R_{sjlp}\\ &\quad + \sum_s T_{ij,s}\,R_{sklp},\end{aligned}$$

$$\begin{aligned}T_{ij,klpm} - T_{ij,klmp} &= \sum_s T_{sj,kl}\,R_{sipm} + \sum_s T_{is,kl}\,R_{sjpm}\\ &\quad + \sum_s T_{ij,sl}\,R_{skpm} + \sum_s T_{ij,ks}\,R_{slpm}.\end{aligned}$$

For the sake of simplicity, we always omit the comma (,) between indices in the special case $T = \sum_{i,j} h_{ij}\,\omega_i\,\omega_j$ with $N^{n+1} = S^{n+1}$.

Since $\sum_{C,D} K_{(n+1)iCD}\,\omega_C \wedge \omega_D = 0$ on M^n when $N^{n+1} = S^{n+1}$, we find

$$d(\sum_j h_{ij}\,\omega_j) = \sum_{j,l} h_{jl}\,\omega_l \wedge \omega_{ji}.$$

Therefore,

$$\sum_{j,l} h_{ijl}\,\omega_l \wedge \omega_j = \sum_j (dh_{ij} + \sum_l h_{lj}\,\omega_{li} + \sum_l h_{il}\,\omega_{lj}) \wedge \omega_j = 0;$$

i.e., h_{ijl} is symmetric in all indices.

In the case M^n is minimal, we have

(1.5)

$$\begin{aligned}
\sum_l h_{ijll} &= \sum_l h_{lijl} = \sum_l \{h_{llij} + \sum_m (h_{mi}R_{mljl} + h_{lm}R_{mijl})\} \\
&= (n-1)h_{ij} \\
&+ \sum_{l,m} \{-h_{mi}h_{ml}h_{lj} + h_{lm}(\delta_{mj}\delta_{il} - \delta_{ml}\delta_{ij} + h_{mj}h_{il} - h_{ml}h_{ij})\} \\
&= nh_{ij} - \sum_{l,m} h_{lm}h_{ml}h_{ij} = (n-S)h_{ij}.
\end{aligned}$$

It follows that

$$\frac{1}{2}\Delta S = (n-S)S + \sum_{i,j,l} h_{ijl}^2. \tag{1.6}$$

2. G-invariant Hypersurface in S^{n+1}.

For $G \simeq O(k) \times O(p) \times O(q)$, $\mathbb{R}^{n+2}$ splits into the orthogonal direct sum of irreducible invariant subspaces, namely

$$\mathbb{R}^{n+2} \simeq \mathbb{R}^k \oplus \mathbb{R}^p \oplus \mathbb{R}^q = \{(X, Y, Z)\}$$

where X is a generic k-vector, Y is a generic p-vector and Z is a generic q-vector. Here if we set $x = |X|, y = |Y|$ and $z = |Z|$, then the orbit space $\mathbb{R}^{n+2}/G$ can be parametrized by (x, y, z); $x, y, z \in \mathbb{R}_+$ and the orbital distance metric is given by $ds^2 = dx^2 + dy^2 + dz^2$. By restricting the above G-action to the unit sphere $S^{n+1} \subset \mathbb{R}^{n+2}$, it is easy to see that

$$S^{n+1}/G \simeq \{(x,y,z) : x^2+y^2+z^2 = 1;\ x,\ y,\ z \geq 0\}$$

which is isometric to a spherical triangle of $S^2(1)$ with $\pi/2$ as its three angles. The orbit labeled by (x,y,z) is exactly $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$. To investigate those *$G$-invariant minimal hypersurfaces*, M^n, in S^{n+1} we study their generating curves, $\gamma(s) = M^n/G$, in the orbit space S^{n+1}/G [4, 6].

The following two lemmas are immediate consequences of Theorem and Corollary, respectively in [6].

LEMMA 2.1. *Let M^n be a G-invariant hypersurface in S^{n+1}. Then there is a local orthonormal frame field $e_1, \ldots, e_{n+1}$ in S^{n+1} such that after restriction to M^n, the $e_1, \ldots, e_n$ are tangent to M^n and $h_{ij} = 0$ if $i \neq j$.*

Proof. Let $(X_0, Y_0, Z_0) \in M^n \subset S^{n+1}$ with $x = |X_0|$, $y = |Y_0|$ and $z = |Z_0|$ and choose a local orthonormal frame field on a neighborhood of (X_0, Y_0, Z_0) as follows.

First, we choose vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ on a neighborhood U of (X_0, Y_0, Z_0) in the orbit $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ such that:

(1) $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}$ are lifts of orthonormal tangent vector fields $u_1, \ldots, u_{k-1}$ on a neighborhood of X_0 in $S^{k-1}(x)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively

(2) $\widetilde{v}_1, \ldots, \widetilde{v}_{p-1}$ are lifts of orthonormal tangent vector fields $v_1, \ldots, v_{p-1}$ on a neighborhood of Y_0 in $S^{p-1}(y)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively,

(3) $\widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ are lifts of orthonormal tangent vector fields $w_1, \ldots, w_{q-1}$ on a neighborhood of Z_0 in $S^{q-1}(z)$ to $S^{k-1}(x) \times S^{p-1}(y) \times S^{q-1}(z)$ respectively.

Second, let $c(t) = (c_1(t), c_2(t), c_3(t))$ be the unit speed geodesic in S^{n+1}/G orthogonal to the curve $\gamma(s) = (x(s), y(s), z(s))$. For each $P = (X, Y, Z) \in U$, let $\widetilde{\gamma}(P, s)$ and $\widetilde{c}(P, t)$ be the horizontal lifts in S^{n+1} of $\gamma(s)$ and $c(t)$ through P respectively. Then we see

$$\widetilde{\gamma}'(P, s) = (x'(s)\frac{X}{x},\ y'(s)\frac{Y}{y},\ z'(s)\frac{Z}{z}), \quad \text{and}$$
$$\widetilde{c}'(P, t) = (c_1'(t)\frac{X}{x},\ c_2'(t)\frac{Y}{y},\ c_3'(t)\frac{Z}{z}).$$

Third, we extend these vector fields over a neighborhood of (X_0, Y_0, Z_0) in S^{n+1} as follows:

(1) we translate $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}$ parallel along $\widetilde{\gamma}$ and $\widetilde{c}$.

(2) we extend $\widetilde{\gamma}'$ and $\widetilde{c}'$ in the usual fashion.

Then these extended vector fields $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}', \widetilde{c}'$ is a local orthonormal frame field in S^{n+1}. After restriction these vector fields to M^n, $\widetilde{u}_1, \ldots, \widetilde{u}_{k-1}, \widetilde{v}_1, \ldots, \widetilde{v}_{p-1}, \widetilde{w}_1, \ldots, \widetilde{w}_{q-1}, \widetilde{\gamma}'$ are tangent to M^n. For convenience, we write them as $e_1, \ldots, e_{n+1}$, in order.

Then, we have

$$\overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{\gamma}' = \frac{x'(0)}{x}\widetilde{u}_i \quad \text{and} \quad \overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{c}' = \frac{c_1'(0)}{x}\widetilde{u}_i, \tag{2.1}$$

and so

$$h_{ij} = \langle \overline{\nabla}_{\widetilde{u}_i(P)}\widetilde{u}_j,\ \widetilde{c}'(0) \rangle = \frac{-c_1'(0)}{x}\delta_{ij} = (\cos r\,\frac{d\theta}{ds} + \frac{\tan\theta}{\sin r}\frac{dr}{ds})\,\delta_{ij}.$$

In the same way, we get

$$h_{ij} = 0 \quad \text{if } i \neq j \text{ for all } 1 \leq i, j \leq n \tag{2.2}$$

and

(2.3)
$$h_{ii} = \begin{cases} \cos r\,(d\theta/ds) + (\tan\theta/\sin r)(dr/ds) & \text{if } 1 \leq i \leq k-1, \\ \cos r\,(d\theta/ds) - (\cot\theta/\sin r)(dr/ds) & \text{if } k \leq i \leq k+p-2, \\ -(\sin^2 r/\cos r)(d\theta/ds) & \text{if } k+p-1 \leq i \leq n-1, \\ (d\alpha/ds) + \cos r(d\theta/ds) & \text{if } i = n, \end{cases}$$

where α is the angle between the curve γ and the radial direction $\partial/\partial r$ (cf. [4, 6]). The proof is complete.

Lemma 2.2. *Let M^n be a G-invariant hypersurface in S^{n+1} and let $\{e_A\}$ be a local orthonormal frame field in S^{n+1} as in Lemma 2.1. Then,*

(a) all $h_{ijl} = 0$ except when $\{i,j,l\}$ is a permutation of either $\{i,i,n\}$,

(b) if $j \neq k$, then $h_{iijk} = 0$, $h_{jkii} = 0$, $h_{jjjk} = 0$ and $h_{kjjj} = 0$,

(c) if i,j,k,l are distinct, then $h_{ijkl} = 0$.

Proof. cf. [6].

Under such frame field as Lemma 2.1, we have

$$(2.4) \qquad e_k(h_{ii}) = h_{iik} - \sum_s h_{si}\omega_{si}(e_k) - \sum_s h_{is}\omega_{si}(e_k) = h_{iik}.$$

Hence, in the case M^n is minimal, by differentiating $\sum_k h_{kk} = 0$ we have

(2.5)

$$\begin{aligned}
0 &= (e_j\, e_i - \nabla_{e_j} e_i)(\sum_k h_{kk}) \\
&= \sum_k \{e_j(h_{kki}) - \sum_s \omega_{is}(e_j)h_{kks}\} \\
&= \sum_k h_{kkij} \\
&\quad - \sum_{k,s} \{h_{ski}\,\omega_{sk}(e_j) + h_{ksi}\,\omega_{sk}(e_j) + h_{kks}\,\omega_{si}(e_j) + h_{kks}\omega_{is}(e_j)\} \\
&= \sum_k h_{kkij}.
\end{aligned}$$

3. G-invariant minimal Hypersurface in S^5.

Throughout this section, we assume that $G \simeq O(2) \times O(2) \times O(2)$ and M^4 is a closed G-invariant minimal hypersurface with constant scalar curvature in S^5. Let $\{e_A\}$ be a local orthonormal frame field in S^5 as in Lemma 2.1. Then by differentiating

$\sum_i h_{ii} = 0$ and $\sum_i h_{ii}^2 = S$ with respect to e_4 respectively, we have

$$h_{114} + h_{224} + h_{334} + h_{444} = 0, \tag{3.1}$$

$$h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334} + h_{44}h_{444} = 0. \tag{3.2}$$

From (2.5), we have

$$h_{11ii} + h_{22ii} + h_{33ii} + h_{44ii} = 0. \tag{3.3}$$

From (1.5), we also have

$$h_{ii11} + h_{ii22} + h_{ii33} + h_{ii44} = (4 - S)h_{ii}. \tag{3.4}$$

Since S is constant, (1.6) and Lemma 2.2 imply

$$S(S-4) = \sum_{i,j,l} h_{ijl}^2 = 3\sum_{i\neq 4} h_{ii4}^2 + h_{444}^2. \tag{3.5}$$

Unlike the C. K. Peng and C. L. Terng [7], for the local orthonormal frame field $\{e_A\}$ in S^5 as in Lemma 2.1, in general $\Delta h_{ij} \neq \sum_k h_{ijkk}$ and $\Delta h_{ijl} \neq \sum_m h_{ijlmm}$.

For example, by Lemma 2.1, Lemma 2.2 and (2.4) we see

$$\begin{aligned} e_k e_k(h_{11}) &= e_k(h_{11k}) \\ &= h_{11kk} - \sum_s h_{s1k}\omega_{s1}(e_k) \\ &\qquad - \sum_s h_{1sk}\omega_{s1}(e_k) - \sum_s h_{11s}\omega_{sk}(e_k). \end{aligned} \tag{3.6}$$

That is,

(3.7)

$$e_1e_1(h_{11}) = h_{1111} - 3h_{114}\omega_{41}(e_1),\ e_2e_2(h_{11}) = h_{1122} - h_{114}\omega_{42}(e_2),$$
$$e_3e_3(h_{11}) = h_{1133} - h_{114}\omega_{43}(e_3),\ e_4e_4(h_{11}) = h_{1144}.$$

Hence, from (2.4) and (3.7) we have

$$\begin{aligned}
(3.8)\qquad \Delta h_{11} &= \sum_k (e_k e_k - \nabla_{e_k} e_k)(h_{11}) \\
&= \sum_k \{e_k e_k - \sum_s \omega_{ks}(e_k)e_s\}(h_{11}) \\
&= \sum_k e_k e_k(h_{11}) + \sum_k h_{114}\omega_{4k}(e_k) \\
&= \sum_k h_{11kk} - 2h_{114}\omega_{41}(e_1) \neq \sum_k h_{11kk}.
\end{aligned}$$

Nevertheless, we also have our Theorem:

THEOREM. *Let $\{e_A\}$ be a local orthonormal frame field in S^5 as in Lemma 2.1. Then, everywhere*

$$(3.9)\qquad \sum_{i,j,l,m} h_{ijlm}^2 = S(S-4)(S-11) + 3(A-2B),$$

where

$$A = \sum_{i,j,k} h_{ijk}^2\, h_{ii}^2 \quad \text{and} \quad B = \sum_{i,j,k} h_{ijk}^2\, h_{ii}\, h_{jj}.$$

Proof. Now, we have

(3.10)

$$\begin{aligned}
e_m(\sum_{i,j,l} h_{ijl}^2) &= 2\sum_{i,j,l} h_{ijl}\, e_m(h_{ijl}) \\
&= 2\sum_{i,j,l} h_{ijl}[h_{ijlm} - \sum_s \{h_{sjl}\,\omega_{si}(e_m) + h_{isl}\omega_{sj}(e_m) + h_{ijs}\omega_{sl}(e_m)\}] \\
&= 2\sum_{i,j,l} h_{ijl}h_{ijlm},
\end{aligned}$$

since

$$\sum_s \{h_{sjl}\,\omega_{si}(e_m) + h_{isl}\omega_{sj}(e_m) + h_{ijs}\omega_{sl}(e_m)\} = 0$$

whenever $h_{ijl} \neq 0$. Hence, we have

(3.11)

$$
\begin{aligned}
\frac{1}{2}\Delta\Big(\sum_{i,j,l} h_{ijl}^2\Big) &= \frac{1}{2}\sum_m (e_m\, e_m - \nabla_{e_m} e_m)\Big(\sum_{i,j,l} h_{ijl}^2\Big) \\
&= \sum_{i,j,l,m} e_m(h_{ijl}h_{ijlm}) - \sum_{i,j,l,m,s} h_{ijl}h_{ijls}\omega_{ms}(e_m) \\
&= \sum_{i,j,l,m} \{(h_{ijlm} - \sum_s h_{sjl}\omega_{si}(e_m) \\
&\quad - \sum_s h_{isl}\omega_{sj}(e_m) - \sum_s h_{ijs}\omega_{sl}(e_m)\}h_{ijlm} \\
&\quad + h_{ijl}\{h_{ijlmm} - \sum_s h_{sjlm}\omega_{si}(e_m) \\
&\quad - \sum_s h_{islm}\omega_{sj}(e_m) - \sum_s h_{ijsm}\omega_{sl}(e_m) \\
&\quad - \sum_s h_{ijls}\omega_{sm}(e_m)\}h_{ijlm} \\
&\quad - \sum_{i,j,l,m,s} h_{ijl}h_{ijls}\omega_{ms}(e_m) \\
&= \sum_{i,j,l,m} (h_{ijlm}^2 + h_{ijl}h_{ijlmm}) \\
&\quad - \sum_{i,j,l,m,s} \{h_{sjl}h_{ijlm}\omega_{si}(e_m) + h_{ijl}h_{sjlm}\omega_{si}(e_m)\} \\
&\quad - \sum_{i,j,l,m,s} \{h_{isl}h_{ijlm}\omega_{sj}(e_m) + h_{ijl}h_{islm}\omega_{sj}(e_m)\} \\
&\quad - \sum_{i,j,l,m,s} \{h_{ijs}h_{ijlm}\omega_{sl}(e_m) + h_{ijl}h_{ijsm}\omega_{sl}(e_m)\} \\
&= \sum_{i,j,l,m} (h_{ijlm}^2 + h_{ijl}h_{ijlmm}).
\end{aligned}
$$

Since S is constant in (3.5), $\sum_{i,j,l} h_{ijl}^2$ is also constant. Hence, (3.11) becomes

$$
\sum_{i,j,l,m} h_{ijlm}^2 = -\sum_m \sum_{i\neq 4} (3h_{ii4}h_{ii4mm} + h_{444}h_{444mm}). \tag{3.12}
$$

Since $\sum_k h_{kkij} = 0$ from (2.5), we have

(3.13)

$$\begin{aligned}\sum_k h_{kkijl} &= e_l(\sum_k h_{kkij}) + \sum_{k,s}\{h_{skij}\,\omega_{sk}(e_l) + h_{ksij}\,\omega_{sk}(e_l) \\ &\qquad\qquad + h_{kksj}\,\omega_{si}(e_l) + h_{kkis}\,\omega_{sj}(e_l)\} \\ &= e_l(\sum_k h_{kkij}) = 0.\end{aligned}$$

To compute the right side of (3.12), from now on, we will use $i,\ j\ \ (i \neq j)$ for indices running over $\{1,\ 2,\ 3\}$.

Since $h_{ii4j} = h_{iij4} = h_{j4ii} = 0$, (1.3), (1.4) and Lemma 2.2 imply

$$\begin{aligned}h_{ii4jj} &= e_j(h_{j4ii}) + h_{iijj}\,\omega_{j4}(e_j) + h_{ii44}\,\omega_{4j}(e_j) \\ &= h_{j4iij} - \{h_{44ii}\,\omega_{4j}(e_j) + h_{jjii}\,\omega_{j4}(e_j)\} \\ &\qquad + h_{iijj}\,\omega_{j4}(e_j) + h_{ii44}\,\omega_{4j}(e_j) \\ &= h_{j4iji} + \sum_s h_{s4i}R_{sjij} + \sum_s h_{jsi}R_{s4ij} + \sum_s h_{j4s}R_{siij} \\ &\quad - \{h_{44ii}\,\omega_{4j}(e_j) + h_{jjii}\,\omega_{j4}(e_j)\} \\ &\qquad + h_{iijj}\,\omega_{j4}(e_j) + h_{ii44}\,\omega_{4j}(e_j) \\ &= e_i(h_{j4ij}) + h_{jiij}\,\omega_{i4}(e_i) + h_{j44j}\,\omega_{4i}(e_i) \\ &\qquad + h_{i4i}R_{ijij} + h_{j4j}R_{jiij} \\ &\quad - \{h_{44ii}\,\omega_{4j}(e_j) + h_{jjii}\,\omega_{j4}(e_j)\} \\ &\qquad + h_{iijj}\,\omega_{j4}(e_j) + h_{ii44}\,\omega_{4j}(e_j) \\ &= h_{jj4ii} + h_{i4i}R_{ijij} + h_{j4j}R_{jiij} \\ &\quad - h_{jjii}\,\omega_{i4}(e_i) - h_{jj44}\,\omega_{4i}(e_i) \\ &\qquad + h_{jiij}\,\omega_{i4}(e_i) + h_{j44j}\,\omega_{4i}(e_i) \\ &\quad - h_{44ii}\,\omega_{4j}(e_j) - h_{jjii}\,\omega_{j4}(e_j) \\ &\qquad + h_{iijj}\,\omega_{j4}(e_j) + h_{ii44}\,\omega_{4j}(e_j)\end{aligned}$$

Here, from (1.1) we have

(3.14)

$$R_{ijij} = K_{ijij} + h_{ii}h_{jj} = 1 + h_{ii}h_{jj} \quad \text{and} \quad R_{jiij} = -1 - h_{jj}h_{ii}.$$

Hence, we have

(3.15)
$$\begin{aligned} h_{ii4jj} - h_{jj4ii} = (1 + h_{ii}h_{jj})h_{ii4} - (1 + h_{ii}h_{jj})h_{jj4} \\ + (h_{jjii} - h_{iijj} + h_{44jj} - h_{jj44})\,\omega_{4i}(e_i) \\ + (h_{jjii} - h_{iijj} - h_{44ii} + h_{ii44})\,\omega_{4j}(e_j). \end{aligned}$$

In the similar way, since $e_4(h_{4i4i}) = e_4(h_{44ii}) = h_{44ii4}$ and $h_{44i4} = 0 = h_{444i}$,

$$\begin{aligned} h_{ii444} &= e_4(h_{ii44}) = e_4(h_{4ii4}) \\ &= e_4\{h_{4i4i} + (h_{ii} - h_{44})(1 + h_{ii}h_{44})\} \\ &= h_{44ii4} + (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) \\ &+ (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\ &= h_{44i4i} + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\ &+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\ &= e_i(h_{44i4}) + h_{i4i4}\,\omega_{i4}(e_i) + h_{4ii4}\,\omega_{i4}(e_i) + h_{4444}\,\omega_{4i}(e_i) \\ &+ h_{44ii}\,\omega_{i4}(e_i) + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\ &+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\ &= h_{444ii} - h_{i44i}\,\omega_{i4}(e_i) - h_{4i4i}\,\omega_{i4}(e_i) - h_{44ii}\,\omega_{i4}(e_i) \\ &- h_{4444}\,\omega_{4i}(e_i) + h_{i4i4}\,\omega_{i4}(e_i) + h_{4ii4}\,\omega_{i4}(e_i) + h_{4444}\,\omega_{4i}(e_i) \\ &+ h_{44ii}\,\omega_{i4}(e_i) + h_{i4i}R_{i4i4} + h_{4ii}R_{i4i4} + h_{444}R_{4ii4} \\ &+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}) \\ &= h_{444ii} + 2(h_{44ii} - h_{ii44})\,\omega_{4i}(e_i) + 2h_{ii4}R_{i4i4} + h_{444}R_{4ii4} \\ &+ (h_{ii4} - h_{444})(1 + h_{ii}h_{44}) + (h_{ii} - h_{44})(h_{ii4}h_{44} + h_{ii}h_{444}). \end{aligned}$$

Hence,

(3.16)
$$\begin{aligned} h_{ii444} - h_{444ii} = (3 + 4h_{ii}h_{44} - h_{44}^2)h_{ii4} - (2 + 3h_{ii}h_{44} - h_{ii}^2)h_{444} \\ + 2(h_{44ii} - h_{ii44})\,\omega_{4i}(e_i). \end{aligned}$$

Here, from (1.3) we know

(3.17)
$$h_{iiii} = 3h_{ii4}\,\omega_{4i}(e_i),\;\; h_{iijj} = h_{ii4}\,\omega_{4j}(e_j),\;\; (h_{44} - h_{ii})\,\omega_{4i}(e_i) = h_{ii4}.$$

Hence, by using (3.1), ..., (3.4), (3.13) and (3.15), ..., (3.17) we get

(3.18)

$$
\begin{aligned}
&3h_{114}(h_{11411}+h_{11422}+h_{11433}+h_{11444})\\
&= 3h_{114}(h_{11411}+h_{22411}+h_{33411}+h_{44411})\\
&\quad + 3h_{114}\{(h_{11422}-h_{22411})+(h_{11433}-h_{33411})\\
&\quad + (h_{11444}-h_{44411})\}\\
&= 3(1+h_{11}h_{22})h_{114}^2 - 3(1+h_{11}h_{22})h_{114}h_{224}\\
&\quad + 3(1+h_{11}h_{33})h_{114}^2 - 3(1+h_{11}h_{33})h_{114}h_{334}\\
&\quad + 3(3+4h_{11}h_{44}-h_{44}^2)h_{114}^2 - 3(2+3h_{11}h_{44}-h_{11})h_{114}h_{444}\\
&\quad + (h_{2211}-h_{1122}+h_{4422}-h_{2244})h_{1111}\\
&\quad + 3(h_{2211}-h_{1122}-h_{4411}+h_{1144})h_{1122}\\
&\quad + (h_{3311}-h_{1133}+h_{4433}-h_{3344})h_{1111}\\
&\quad + 3(h_{3311}-h_{1133}-h_{4411}+h_{1144})h_{1133} + 2(h_{4411}-h_{1144})h_{1111}\\
&= 15h_{114}^2 + 3(h_{22}+h_{33}+h_{44})h_{11}h_{114}^2 - 3h_{114}(h_{224}+h_{334}+h_{444})\\
&\quad - 3h_{11}h_{114}(h_{22}h_{224}+h_{33}h_{334}+h_{44}h_{444})\\
&\quad + 9h_{44}h_{11}h_{114}^2 - 3h_{44}^2h_{114}^2 - 3h_{114}h_{444}\\
&\quad - 6h_{11}h_{114}h_{44}h_{444} + 3h_{11}^2h_{114}h_{444}\\
&\quad + (h_{1111}+h_{2211}+h_{3311}+h_{4411})h_{1111}\\
&\quad - (h_{1111}+h_{1122}+h_{1133}+h_{1144})h_{1111}\\
&\quad + (h_{4411}+h_{4422}+h_{4433}+h_{4444})h_{1111}\\
&\quad - (h_{1144}+h_{2244}+h_{3344}+h_{4444})h_{1111}\\
&\quad + 3(h_{2211}-h_{1122}-h_{4411}+h_{1144})h_{1122}\\
&\quad + 3(h_{3311}-h_{1133}-h_{4411}+h_{1144})h_{1133}\\
&= 18h_{114}^2 + 9h_{11}h_{44}h_{114}^2 - 3h_{44}^2h_{114}^2 - 3h_{114}h_{444}\\
&\quad - 6h_{11}h_{114}h_{44}h_{444} + 3h_{11}^2h_{114}h_{444} + 3(4-S)h_{114}^2\\
&\quad + 3(h_{2211}-h_{1122}-h_{4411}+h_{1144})h_{1122}\\
&\quad + 3(h_{3311}-h_{1133}-h_{4411}+h_{1144})h_{1133}.
\end{aligned}
$$

In the same way, we have

$$
\begin{aligned}
&3h_{224}(h_{22411}+h_{22422}+h_{22433}+h_{22444})\\
&=18h_{224}^2+9h_{22}h_{44}h_{224}^2-3h_{44}^2h_{224}^2\\
&\quad-3h_{224}h_{444}-6h_{22}h_{224}h_{44}h_{444}\\
&\quad+3h_{22}^2h_{224}h_{444}+3(4-S)h_{224}^2\\
&\quad+3(-h_{2211}+h_{1122}-h_{4422}+h_{2244})h_{2211}\\
&\quad+3(h_{3322}-h_{2233}-h_{4422}+h_{2244})h_{2233},
\end{aligned}
\tag{3.19}
$$

and

$$
\begin{aligned}
&3h_{334}(h_{33411}+h_{33422}+h_{33433}+h_{33444})\\
&=18h_{334}^2+9h_{33}h_{44}h_{334}^2-3h_{44}^2h_{334}^2\\
&\quad-3h_{334}h_{444}-6h_{33}h_{334}h_{44}h_{444}\\
&\quad+3h_{33}^2h_{334}h_{444}+3(4-S)h_{334}^2\\
&\quad-3(h_{3311}-h_{1133}+h_{4433}-h_{3344})h_{3311}\\
&\quad-3(h_{3322}-h_{2233}+h_{4433}-h_{3344})h_{3322}.
\end{aligned}
\tag{20}
$$

Moreover, since

$$h_{44ii}-h_{ii44}=(h_{44}-h_{ii})(1+h_{44}h_{ii}),\ \ (h_{44}-h_{ii})\,\omega_{4i}(e_i)=h_{ii4},$$

we have

(3.21)
$$
\begin{aligned}
&h_{444}(h_{44411}+h_{44422}+h_{44433}+h_{44444})\\
&= h_{444}(h_{11444}+h_{22444}+h_{33444}+h_{44444})\\
&+ h_{444}\{(h_{44411}-h_{11444})+(h_{44422}-h_{22444})+(h_{44433}-h_{33444})\}\\
&= -(3+4h_{11}h_{44}-h_{44}^2)h_{114}h_{444}+(2+3h_{11}h_{44}-h_{11}^2)h_{444}^2\\
&\quad -(3+4h_{22}h_{44}-h_{44}^2)h_{224}h_{444}+(2+3h_{22}h_{44}-h_{22}^2)h_{444}^2\\
&\quad -(3+4h_{33}h_{44}-h_{44}^2)h_{334}h_{444}+(2+3h_{33}h_{44}-h_{33}^2)h_{444}^2\\
&\quad -2(h_{4411}-h_{1144})h_{444}\,\omega_{41}(e_1)-2(h_{4422}-h_{2244})h_{444}\,\omega_{42}(e_2)\\
&\quad -2(h_{4433}-h_{3344})h_{444}\,\omega_{43}(e_3)\\
&= -5(h_{114}+h_{224}+h_{334})h_{444}\\
&\quad -6(h_{11}h_{114}+h_{22}h_{224}+h_{33}h_{334})h_{44}h_{444}\\
&\quad +(h_{114}+h_{224}+h_{334})h_{44}^2h_{444}\\
&\quad +6h_{444}^2+3(h_{11}+h_{22}+h_{33})h_{44}h_{444}^2-(S-h_{44}^2)h_{444}^2\\
&= (11-S)h_{444}^2+3h_{44}^2h_{444}^2.
\end{aligned}
$$

Here,

(3.22)
$$
\begin{aligned}
3(A-2B) &= -12(h_{114}^2h_{11}h_{44}+h_{224}^2h_{22}h_{44}+h_{334}^2h_{33}h_{44})\\
&\quad +3(h_{114}^2+h_{224}^2+h_{334}^2)h_{44}^2-3h_{444}^2h_{44}^2.
\end{aligned}
$$

Hence, from (3.18), …, (3.22) we obtain

(3.23)
$$
\begin{aligned}
&3\sum_{i,m} h_{ii4}h_{ii4mm}+\sum_m h_{444}h_{444mm}\\
&\quad = S(S-4)(11-S)-3(A-2B)-S(S-4)\\
&\qquad -3\sum_i h_{ii4}^2h_{ii}h_{44}+3\sum_i h_{ii}^2h_{ii4}h_{444}+6h_{444}^2h_{44}^2+4h_{444}^2\\
&\qquad -3\sum_{i<j}(h_{iijj}-h_{jjii})^2-3\sum_{i\neq j}(h_{44ii}-h_{ii44})h_{iijj}.
\end{aligned}
$$

On the other hand, since $h_{jjii}=h_{jj4}\,\omega_{4i}(e_i)$, $(h_{44}-h_{ii})\,\omega_{4i}(e_i)$

$= h_{ii4}$, we have

(3.24)
$$\begin{aligned}(h_{44ii} - h_{ii44})h_{iijj} &= (h_{44ii} - h_{ii44})h_{jjii} \\ &\quad + (h_{44ii} - h_{ii44})(h_{iijj} - h_{jjii}) \\ &= (1 + h_{ii}h_{44})h_{ii4}h_{jj4} \\ &\quad + (h_{44} - h_{ii})(1 + h_{ii}h_{44})(h_{ii} - h_{jj})(1 + h_{ii}h_{jj}).\end{aligned}$$

Now, we know

(3.25)
$$\begin{aligned}&-3\sum_{i\neq j}(1 + h_{ii}h_{44})h_{ii4}h_{jj4} \\ &= -3(h_{114} + h_{224} + h_{334})^2 + 3h_{114}^2 + 3h_{224}^2 + 3h_{334}^2 \\ &\quad - 3(h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334})(h_{114} + h_{224} + h_{334})h_{44} \\ &\quad + 3(h_{114}^2h_{11}h_{44} + h_{224}^2h_{22}h_{44} + h_{334}^2h_{33}h_{44}) \\ &= S(S-4) - 4h_{444}^2 - 3h_{444}^2h_{44}^2 + 3\sum_i h_{ii4}^2h_{ii}h_{44},\end{aligned}$$

and

(3.26)
$$\begin{aligned}&\sum_{i<j}(h_{iijj} - h_{jjii})^2 \\ &\qquad + \sum_{i\neq j}(h_{44} - h_{ii})(1 + h_{ii}h_{44})(h_{ii} - h_{jj})(1 + h_{ii}h_{jj}) \\ &= \sum_{i<j}(h_{ii} - h_{jj})^2(h_{44} - h_{ii})(h_{44} - h_{jj})(1 + h_{ii}h_{jj}).\end{aligned}$$

Hence, (3.25) and (3.26) imply that

(3.27)
$$\begin{aligned}&-3\sum_{i<j}(h_{iijj} - h_{jjii})^2 - 3\sum_{i\neq j}(h_{44ii} - h_{ii44})h_{iijj} \\ &\qquad = S(S-4) - 4h_{444}^2 - 3h_{444}^2h_{44}^2 + 3\sum_i h_{ii4}^2h_{ii}h_{44} \\ &\qquad\quad - 3\sum_{i<j}(h_{ii} - h_{jj})^2(h_{44} - h_{ii})(h_{44} - h_{jj})(1 + h_{ii}h_{jj}).\end{aligned}$$

By substituting the equality (3.27) into (3.23), we have

(3.28)
$$
\begin{aligned}
3\sum_{i,m} h_{ii4}h_{ii4mm} + \sum_{m} h_{444}h_{444mm}
&= S(S-4)(11-S) - 3(A-2B) + 3\sum_i h_{ii}^2 h_{ii4}h_{444} + 3h_{444}^2 h_{44}^2 \\
&\quad - 3\sum_{i<j}(h_{ii}-h_{jj})^2(h_{44}-h_{ii})(h_{44}-h_{jj})(1+h_{ii}h_{jj}).
\end{aligned}
$$

Now, by using $(h_{44}-h_{ii})\,\omega_{4i}(e_i) = h_{ii4}$, $h_{ii4}\,\omega_{4j}(e_j) = h_{iijj}$ we have

$$
(3.29)\quad h_{ii4}h_{jj4} = (h_{44}-h_{ii})h_{jjii} \quad\text{and}\quad h_{jj4}h_{ii4} = (h_{44}-h_{jj})h_{iijj}.
$$

From (3.29), we have

$$
(3.30)\qquad
\begin{aligned}
h_{jj4}h_{ii4} &= (h_{44}-h_{jj})h_{iijj} \\
&= (h_{44}-h_{jj})\{h_{jjii} + (h_{ii}-h_{jj})(1+h_{ii}h_{jj})\} \\
&= (h_{44}-h_{ii})h_{jjii}.
\end{aligned}
$$

Hence, from (3.30) it follows that if $h_{ii} \neq h_{jj}$, then

$$
(3.31)\qquad h_{jjii} = (h_{jj}-h_{44})(1+h_{ii}h_{jj}).
$$

(3.29) and (3.31) give

(3.32)
$$
(h_{ii}-h_{jj})^2 h_{ii4}h_{jj4} = (h_{ii}-h_{jj})^2(h_{44}-h_{ii})(h_{jj}-h_{44})(1+h_{ii}h_{jj}).
$$

Hence, by using (3.1), (3.2) and (3.32), we have

(3.33)
$$
\begin{aligned}
3\sum_i h_{ii}^2 h_{ii4}h_{444} + 3h_{44}^2 h_{444}^2
&= -3(h_{11}^2 h_{114} + h_{22}^2 h_{224} + h_{33}^2 h_{334})(h_{114}+h_{224}+h_{334}) \\
&\quad + 3(h_{11}h_{114} + h_{22}h_{224} + h_{33}h_{334})^2 \\
&= -3\sum_{i<j}(h_{ii}-h_{jj})^2 h_{ii4}h_{jj4} \\
&= 3\sum_{i<j}(h_{ii}-h_{jj})^2(h_{44}-h_{ii})(h_{44}-h_{jj})(1+h_{ii}h_{jj}).
\end{aligned}
$$

Hence, by substituting the equality (3.33) into (3.28) we obtain

$$\begin{aligned}\sum_{i,j,l,m} h_{ijlm}^2 &= -3\sum_{i,m} h_{ii4}h_{ii4mm} - \sum_m h_{444}h_{444mm} \\ &= S(S-4)(S-11) + 3(A-2B),\end{aligned}$$

which completes the proof of Theorem.

References

[1] E. Cartan, *Sur des familles remarquables d'hypersurfaces isoparametriques dans les espaces spheriques*, Math. Z. **45** (1939), 335–367.
[2] S. Chang, *On minimal hypersurfaces with constant scalar curvatures in* S^4, J.Diff.Geom. **37** (1993), 523–534.
[3] S. S. Chern, M. do Carmo & S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Duke Math. J. **61** (1990), 195–206.
[4] W. Y. Hsiang, *On the construction of infinitely many congruence classes of imbedded closed minimal hypersurfaces in* $S^n(1)$ *for all* $n \geq 3$ *Math. J.*.
[5] H. B. Lawson, *Local rigidity theorems for minimal hypersurfaces*, Annals of Math. (2) **89** (1969), 187–191.
[6] H. B. Park, J. H. Park & J. U. So, *On scalar curvatures of G-invariant minimal hypersurfaces in* S^{n+1}, Korean Ann. of Math. **17** (2000), 247–260.
[7] C. K. Peng & C. L. Terng, *Minimal hypersurface of spheres with constant scalar curvature*, Annals of Math. Studies, No. 103, Princeton University Press, Princeton, NJ, (1983), 177–198.
[8] J. Simons, *Minimal varieties in a Riemannian manifold*, Ann. of Math. (2) **88** (1968), 62–105.
[9] S. T. Yau, *Problem section*, Annals of Math. Studies, No. 102, Princeton University Press, Princeton, NJ, (1982), 693.