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Riemann Solvers in Relativistic Hydrodynamics: Basics and Astrophysical Applications

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ABSTRACT

My contribution to these proceedings summarizes a general overview on *High Resolution Shock Capturing methods* (HRSC) in the field of relativistic hydrodynamics with special emphasis on Riemann solvers. HRSC techniques achieve highly accurate numerical approximations (formally second order or better) in smooth regions of the flow, and capture the motion of unresolved steep gradients without creating spurious oscillations. In the first part I will show how these techniques have been extended to relativistic hydrodynamics, making it possible to explore some challenging astrophysical scenarios. I will review recent literature concerning the main properties of different *special relativistic Riemann solvers*, and discuss several 1D and 2D test problems which are commonly used to evaluate the performance of numerical methods in relativistic hydrodynamics. In the second part I will illustrate the use of HRSC methods in several astrophysical applications where special and general relativistic hydrodynamical processes play a crucial role.

Key Words : Numerical Methods: Hyperbolic Systems of Conservation Laws – Relativity – Jets – Accretion – Gamma-Ray Bursts

I. INTRODUCTION

In my two talks I tried to offer to the audience (mainly enthousiastic young korean astrophysicists) a general overview on the present status of: i) *High Resolution Shock Capturing methods* (HRSC) in the field of relativistic hydrodynamics, with special emphasis on schemes based upon Riemann solvers, and ii) their use in astrophysical applications: relativistic jets (in some active galactic nuclei), wind accretion onto black holes (relativistic Bondi-Hoyle-Lyttleton scenario), models of formation of gamma-ray bursts (relativistic jets from collapsars) and stellar core collapse.

Readers interested in deepening into the contents of my talks are addressed to the following reviews: Ibáñez & Martí 1999, Martí & Müller 1999, Font 2000 and Aloy & Martí 2001.

Astrophysical scenarios involving relativistic flows have drawn the attention and efforts of many researchers since the pioneering studies of May & White (1967) and Wilson (1972). Relativistic jets, accretion onto compact objects (in X-ray binaries or in the inner regions of active galactic nuclei), stellar core collapse, coalescing compact binaries (neutron star and/or black holes) and recent models of formation of gamma-ray bursts (GRBs) are examples of systems in which the evolution of matter is described within the framework of the theory of relativity (special or general).

Since 1991 (Martí, Ibáñez & Miralles 1991) the use of Riemann solvers, i.e., algorithms designed to solve Riemann problems (see definition, below) in computational relativistic hydrodynamics has proved succesful in handling complex flows, with high Lorentz factors and strong shocks, superseding traditional methods which failed to describe ultrarelativistic flows (Norman & Winkler 1986). By exploiting the hyperbolic and conservative character of the relativistic hydrodynamic equations, and following the approach developed in Newtonian hydrodynamics, we extended HRSC methods to the relativistic case, first in one-dimensional calculations (Martí, Ibáñez & Miralles 1991), and, later on, in multidimensional special relativistic (Font *et al.* 1994, Donat*et al.* 1998) and multidimensional general relativistic hydrodynamics (Banyuls *et al.* 1997, Ibáñez *et al.* 2001). Our approach made use of a linearized Riemann solver based on the knowledge of the *spectral decomposition* of the Jacobian matrices of the system.

Unlike the case of classical fluid dynamics the use of HRSC techniques in relativistic fluid dynamics is very recent and has yet to cover the full set of possible applications. In the second part of this proceedings I will summarize some of the most recent applications in modelling relativistic astrophysical systems.

The task of developing robust, stable and accurate (special or general) relativistic hydrocodes is a challenge in the field of Relativistic Astrophysics. A general relativistic hydrocode is a useful research tool for studying flows which evolve in a background spacetime. Furthermore, when appropiately coupled with Einstein equations, such a general relativistic hydrocode is crucial to model the evolution of matter in a dynamical spacetime. The coupling between geometry and matter arises through the sources of the corresponding system of equations. Such a marriage between numerical relativity and numerical relativistic hydrodynamics is useful, for example, to analyze the dynamics (and the physics) of coalescing compact binaries. These are one of the most promising sources of gravitational radiation to be detected by the near future Earth-based laser-interferometer observatories of gravitational waves.

II. HYPERBOLIC SYSTEM OF CONSERVATION LAWS

For the sake of consistency, in this Section I summarize the basic definitions and properties of hyperbolic systems of conservation laws (subsection §*II*.1), in connection with HRSC techniques (subsection §*II*.2), and applied them to the particular system of equations of relativistic hydrodynamics (subsections §*II*.3 and §*II*.4). Further mathematical details can be found in the following textbooks: Anile (1989), Lax (1972), Leveque (1991) and Toro (1997).

II.1.-Hyperbolic systems of conservation laws: Basics

Let us start by considering the system of p equations of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{d} \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial x_j} = 0 \ \ (= \mathbf{s}(\mathbf{u})) \tag{1}$$

where $\mathbf{u} = (u_1, u_2, \ldots, u_p)^T$ is the vector of unknowns, function of $\mathbf{x}$ and t, with $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in R^d$ and $\mathbf{f}_j(\mathbf{u}) = (f_{1j}, f_{2j}, \ldots, f_{pj})^T$ is the vector of fluxes.

Formally, system (1) expresses the conservation of the vector $\mathbf{u}$. Let D be an arbitrary domain of R^d and let $\mathbf{n} = (n_1, \ldots, n_d)$ be the outward unit normal to the boundary ∂D of D. Then, from (1), it follows that

$$\frac{d}{dt}\int_D \mathbf{u}\, d\mathbf{x} + \sum_{j=1}^{d}\int_{\partial D} \mathbf{f}_j(\mathbf{u}) n_j \, dS = 0. \tag{2}$$

This balance equation establishes that the time variation of $\int_D \mathbf{u}\, d\mathbf{x}$ is equal to the losses through the boundary ∂D.

Now, for all $j = 1, \ldots, d$ let

$$\mathbf{A}_j(\mathbf{u}) = \frac{\partial \mathbf{f}_j(\mathbf{u})}{\partial \mathbf{u}} \tag{3}$$

be the Jacobian matrix of $\mathbf{f}_j(\mathbf{u})$. The system (1) is called *hyperbolic* if, for any $\omega = (\omega_1, \ldots, \omega_d) \in R^d$, and for any $\mathbf{u}$, the matrix

$$\mathbf{A}(\mathbf{u}, \omega) = \sum_{j=1}^{d} \omega_j \mathbf{A}_j(\mathbf{u}) \tag{4}$$

has p real eigenvalues $\lambda_1(\mathbf{u}, \omega) \le \lambda_2(\mathbf{u}, \omega) \le \cdots \le \lambda_p(\mathbf{u}, \omega)$ and p linearly independent (right) eigenvectors $\mathbf{r}_1(\mathbf{u}, \omega)$, $\mathbf{r}_2(\mathbf{u}, \omega)$, ..., $\mathbf{r}_p(\mathbf{u}, \omega)$. If, in addition, the eigenvalues $\lambda_k(\mathbf{u}, \omega)$ are all different, the system (1) is called *strictly hyperbolic.*

In most of the cases one shall be concerned with the so-called *initial value problem* (IVP), i.e., the solution of system (1) with the initial condition

$$\mathbf{u}(\mathbf{x}, t = 0) = \mathbf{u}_0(\mathbf{x}). \tag{5}$$

A key property of hyperbolic systems is that features in the solution propagate at *characteristic speeds* given by the eigenvalues of the Jacobian matrices. The characteristic curves associated to system (1) are the integral curves of the differential equations

$$\frac{dx}{dt} = \lambda_k(\mathbf{u}(x,t)), \quad k = 1, \ldots, p, \tag{6}$$

($d = 1$). It can easily be proven that, along these curves the so-called *characteristic variables* (a combination of the components of $\mathbf{u}$) are constant. Essentially, characteristic curves give information about the propagation of the initial data, which formally allows one to reconstruct the solution for the initial value problem (1) with (5) at $t > 0$.

Continuous and differentiable solutions that satisfy (1) and (5) pointwise are called *classical solutions.* However, for nonlinear systems, classical solutions do not exist in general even when the initial condition $\mathbf{u}_0$ is a smooth function, and discontinuities develop after a finite time. Then one seeks generalized solutions that satisfy the integral form of the conservation system (2) which are classical solutions where they are continuous and have a finite number of discontinuities (*weak solutions*). The following theorem characterizes these solutions.

Let $\mathbf{u}$ be a piecewise smooth function. Then, $\mathbf{u}$ is a solution of the integral form of the conservation system if and only if the two following conditions are satisfied:

1. $\mathbf{u}$ is a classical solution in the domains where it is continuous.

2. Across a given surface of discontinuity, Σ, it satisfies the jump conditions (Rankine-Hugoniot conditions)

$$(\mathbf{u}_R - \mathbf{u}_L) n_t + \sum_{j=1}^{d} [\mathbf{f}_j(\mathbf{u}_R) - \mathbf{f}_j(\mathbf{u}_L)]\, n_{xj} = 0, \tag{7}$$

where $\mathbf{u}_L$ and $\mathbf{u}_R$ stand, respectively, for the values of $\mathbf{u}$ on the left and right hand sides of Σ, and $\mathbf{n} = (n_t, n_{x1}, n_{x2}, \ldots, n_{xd})$ denotes a vector normal to Σ.

For 1D systems, the *Rankine-Hugoniot jump condition* (7) reduces to

$$s(\mathbf{u}_R - \mathbf{u}_L) = \mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L) \tag{8}$$

where s is the *propagation velocity of the discontinuity.*

Rankine-Hugoniot conditions follow from the conservation of fluxes across the surfaces of discontinuity. They can be used in combination with standard finite-difference methods for the smooth regions and special procedures for tracking the location of discontinuities to solve the equations in the presence of shocks (*shock-tracking* approach). In 1D this is often a viable approach. However, in multidimensional applications the discontinuities lie along curves (in 2D) or surfaces (in 3D) and in realistic problems there may be many such discontinuities interacting in complicated ways, making the use of shock-tracking methods much more difficult.

The class of all weak solutions is too wide in the sense that there is no uniqueness for the initial value problem. Therefore, an effort should be made to develop numerical methods picking up the physically admissible solution. Mathematically, this solution is characterized by the so-called *entropy condition* (in the language of fluids, the condition that the entropy of any fluid element should increase when running into a discontinuity). The characterization of the *entropy-satisfying solutions* for scalar equations follows Oleinik (1963), whereas for systems of conservation laws it has been developed by Lax (1972) .

Most HRSC methods are based on exact or approximate solutions of Riemann problems between contiguous numerical cells. Consider the hyperbolic system of conservation laws in 1D

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \tag{9}$$

with initial data $\mathbf{u}(x,0) = \mathbf{u}_0(x)$. A Riemann problem for system (9) is an initial value problem with discontinuous data, i.e.,

$$\mathbf{u}_0 = \begin{cases} \mathbf{u}_L & \text{if } x < 0 \\ \mathbf{u}_R & \text{if } x > 0 \end{cases} \tag{10}$$

The Riemann problem is invariant under similarity transformations $(x,t) \to (ax, at)$, $a > 0$, so that the solution is constant along the straight lines $x/t =$ constant and, hence, self-similar. It consists of constant states separated by *rarefaction waves* (continuous self-similar solutions of the differential system), *shocks* and *contact discontinuities* (Lax 1972). In the following I will denote the *Riemann solution* for the left and right states $\mathbf{u}_L$ and $\mathbf{u}_R$, respectively, as $\mathbf{u}(x/t; \mathbf{u}_L, \mathbf{u}_R)$.

II.2.- High-Resolution Shock-Capturing schemes

Let us start by considering an IVP for (9). Finite-difference methods are based on a discretization of the $x-t$ plane defined by the discrete mesh points (x_j, t^n)

$$x_j = (j - 1/2)\Delta x, \quad j = 1, 2, \ldots \tag{11}$$

$$t^n = n\Delta t, \quad n = 0, 1, 2, \ldots, \tag{12}$$

where Δx and Δt are, respectively the cell width and the time step. A finite-difference scheme is a time-marching procedure allowing one to obtain approximations to the solution in the mesh points, $\mathbf{u}_j^{n+1}$, from the approximations in previous time steps $\mathbf{u}_j^n$. Quantity $\mathbf{u}_j^n$ is an approximation to $\mathbf{u}(x_j, t^n)$ but, in the case of a conservation law, it is often preferable to view it as an approximation to the average of $\mathbf{u}(x,t)$ within the numerical cell $[x_{j-1/2}, x_{j+1/2}]$ $(x_{j\pm 1/2} = (x_j + x_{j\pm 1})/2)$

$$\mathbf{u}_j^n \approx \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{u}(x, t^n) dx, \tag{13}$$

consistent with the integral form of the conservation law.

For hyperbolic systems of conservation laws, methods in *conservation form* are preferred as they guarantee that the convergence (if it exists) is to one of the weak solutions of the original system of equations (*Lax-Wendroff theorem* (Lax & Wendroff 1960). Conservation form means that the algorithm is written as

$$\begin{aligned} \mathbf{u}_j^{n+1} \;\; = \;\; & \mathbf{u}_j^n - \frac{\Delta t}{\Delta x} \Big(\hat{\mathbf{f}}(\mathbf{u}_{j-r}^n, \mathbf{u}_{j-r+1}^n, \ldots, \mathbf{u}_{j+q}^n) - \\ & - \hat{\mathbf{f}}(\mathbf{u}_{j-r-1}^n, \mathbf{u}_{j-r}^n, \ldots, \mathbf{u}_{j+q-1}^n) \Big) \end{aligned} \tag{14}$$

where $\hat{\mathbf{f}}$ is a consistent (i.e., $\hat{\mathbf{f}}(\mathbf{u}, \mathbf{u}, \ldots, \mathbf{u}) = \mathbf{f}(\mathbf{u})$) *numerical flux function.* The Lax-Wendroff theorem does not state whether the method converges. To guarantee convergence, some form of stability is required, as for linear problems (*Lax equivalence theorem* (Richtmyer & Morton 1967). In this direction, the notion of *total-variation stability* has proven very successful although powerful results have only been obtained for scalar conservation laws. The total variation of a solution at $t = t^n$, $\mathrm{TV}(\mathbf{u}^n)$, is defined as

$$\mathrm{TV}(\mathbf{u}^n) = \sum_{j=0}^{+\infty} |\mathbf{u}_{j+1}^n - \mathbf{u}_j^n| \tag{15}$$

and a numerical scheme is said to be TV-stable if $\mathrm{TV}(\mathbf{u}^n)$ is bounded for all Δt at any time for each initial data. For a non-linear scalar conservation law, TV-stability is a sufficient condition for convergence of numerical schemes in conservation form with consistent numerical flux functions (Leveque 1992).

In recent years a very interesting line of research has focused on developing high-order, accurate methods in conservation form satisfying the condition of TV-stability. The conservation form is ensured by starting with the integral version of the partial differential equation in conservation form. Integrating the PDE in a spacetime computational cell $[x_{j-1/2}, x_{j+1/2}] \times [t^n, t^{n+1}]$ and comparing with (14), the numerical flux function $\hat{\mathbf{f}}_{j+1/2}$ is seen to be an approximation to the time-averaged flux across the interface, i.e.,

$$\hat{\mathbf{f}}_{j+1/2} \approx \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \mathbf{f}(\mathbf{u}(x_{j+1/2}, t)) dt. \tag{16}$$

In the above expression, the flux integral depends on the solution at the numerical interfaces, $\mathbf{u}(x_{j+1/2}, t)$, during the time step. Hence, a possible procedure is to calculate $\mathbf{u}(x_{j+1/2}, t)$ by solving Riemann problems at every numerical interface to obtain

$$\mathbf{u}(x_{j+1/2}, t) = \mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n). \tag{17}$$

This is the approach followed by an important subset of shock-capturing methods, the so-called *Godunov-type* *methods* (Harten, Lax & van Leer 1983, Einfeldt 1988). These methods are written in conservation form and use different procedures (Riemann solvers) to compute approximations to $\mathbf{u}(0; \mathbf{u}_j^n, \mathbf{u}_{j+1}^n)$.

High-order of accuracy is usually achieved by using conservative polynomial functions to interpolate the approximate solutions within the numerical cells. The idea is to produce more accurate left and right states for the Riemann problems by substituting the mean values $\mathbf{u}_j^n$ (that give only first-order accuracy) for better approximations of the true flux near the interfaces, $\mathbf{u}_{j+1/2}^{\mathrm{L}}$, $\mathbf{u}_{j+1/2}^{\mathrm{R}}$ (the *flux-corrected-transport* algorithm (Boris & Book 1973) constitutes an alternative procedure where higher accuracy is obtained by adding an anti-diffusive flux term to the first-order numerical flux). The interpolation algorithms have to preserve the TV-stability of the scheme and this is usually achieved by using monotonic functions which lead to a decrease in the total variation (*total-variation-diminishing schemes*, TVD; see Harten 1984). If R is an interpolant function for the approximate solution $\mathbf{u}^n$ and $\tilde{\mathbf{u}}(x, t^n)$ is the interpolated function within the cells, i.e., $\tilde{\mathbf{u}}(x, t^n) = \mathrm{R}(\mathbf{u}^n; x)$, satisfying $\mathrm{TV}(\tilde{\mathbf{u}}(\cdot, t^n)) \leq \mathrm{TV}(\mathbf{u}^n)$ then it can be proven that the whole scheme verifies $\mathrm{TV}(\mathbf{u}^{n+1}) \leq \mathrm{TV}(\mathbf{u}^n)$. High-order TVD schemes were first constructed by van Leer (1979) who obtained second-order accuracy by using monotonic *piecewise linear* slopes for cell reconstruction. The *piecewise parabolic method* (PPM) of Colella and Woodward (1984) provides higher accuracy. The TVD property implies TV-stability but can be too restrictive. In fact, TVD methods degenerate to first order accuracy at extreme points (Osher & Chakravarthy 1984). Hence, other reconstruction alternatives have been developed in which some growth of the total variation is allowed. This is the case of the *total-variation-bounded* schemes (Shu 1987), *essentially nonoscillatory* (ENO) schemes (Harten *et al.* 1987) and the *piecewise-hyperbolic method* (Marquina 1994).

III. THE EQUATIONS OF GENERAL RELATIVISTIC HYDRODYNAMICS AS A HYPERBOLIC SYSTEM OF CONSERVATION LAWS

III.1.- The equations of general relativistic hydrodynamics

The evolution of a relativistic fluid is governed by a system of equations which summarize *local conservation laws*: the local conservation of baryon number, $\nabla \cdot \mathbf{J} = 0$, and the local conservation of energy-momentum, $\nabla \cdot \mathbf{T} = 0$ ($\nabla\cdot$ stands for the covariant divergence).

If $\{\partial_t, \partial_i\}$ define the coordinate basis of 4-vectors which are tangents to the corresponding coordinate curves, then, the *current of rest-mass*, $\mathbf{J}$, and the *energy-momentum tensor*, $\mathbf{T}$, for a perfect fluid, have the components: $J^\mu = \rho u^\mu$, and $T^{\mu\nu} = \rho h u^\mu u^\nu + p g^{\mu\nu}$, respectively, ρ being the rest-mass density, p the pressure and h the specific enthalpy, defined by $h = 1 + \varepsilon + p/\rho$, where ε is the specific internal energy. u^μ is the four-velocity of the fluid and $g_{\mu\nu}$ defines the metric of the spacetime $\mathcal{M}$ where the fluid evolves. As usual, Greek (Latin) indices run from 0 to 3 (1 to 3) – or, alternatively, they stand for the general coordinates $\{t, x, y, z\}$ ($\{x, y, z\}$) – and the system of units is the so-called geometrized ($c = G = 1$).

An equation of state $p = p(\rho, \varepsilon)$ closes, as usual, the system. Accordingly, the local sound velocity c_s satisfies: $hc_s^2 = \chi + (p/\rho^2)\kappa$, with $\chi = \partial p/\partial\rho|_\epsilon$ and $\kappa = \partial p/\partial\epsilon|_\rho$.

Following Banyuls *et al.* (1997), let $\mathcal{M}$ be a general spacetime described by the four dimensional metric tensor $g_{\mu\nu}$. According to the $\{3+1\}$ formalism, the metric is split into the objects α (*lapse*), β^i (*shift*) and γ_{ij}, keeping the line element in the form:

$$ds^2 = -(\alpha^2 - \beta_i\beta^i)dt^2 + 2\beta_i dx^i dt + \gamma_{ij} dx^i dx^j \quad (18)$$

If $\mathbf{n}$ is a unit timelike vector field normal to the spacelike hypersurfaces Σ_t (t = const.), then, by definition of α and β^i is: $\partial_t = \alpha\mathbf{n} + \beta^i\partial_i$, with $\mathbf{n}\cdot\partial_i = 0$, $\forall i$. Observers, $\mathcal{O}_E$, at rest in the slice Σ_t, i.e., those having $\mathbf{n}$ as four-velocity (*Eulerian observers*), measure the following velocity of the fluid

$$v^i = \frac{u^i}{\alpha u^t} + \frac{\beta^i}{\alpha} \quad (19)$$

where $W \equiv -(\mathbf{u}\cdot\mathbf{n}) = \alpha u^t$, the Lorentz factor, satisfies $W = (1 - \mathrm{v}^2)^{-1/2}$ with $\mathrm{v}^2 = v_i v^i$ ($v_i = \gamma_{ij} v^j$).

Let us define a basis adapted to the observer $\mathcal{O}_E$, $\mathbf{e}_{(\mu)} = \{\mathbf{n}, \partial_i\}$, and the following five four-vector fields $\{\mathbf{J}, \mathbf{T}\cdot\mathbf{n}, \mathbf{T}\cdot\partial_1, \mathbf{T}\cdot\partial_2, \mathbf{T}\cdot\partial_3\}$. Hence, the system of equations of general relativistic hydrodynamics can be written

$$\nabla \cdot \mathbf{A} = s, \quad (20)$$

where $\mathbf{A}$ denotes any of the above 5 vector fields, and s is the corresponding source term.

The set of *conserved variables* gathers those quantities which are directly measured by $\mathcal{O}_E$, i.e., the rest-mass density (D), the momentum density in the j-direction (S_j) and the total energy density (E). In terms of the *primitive variables* $\mathbf{w} = (\rho, v_i, \varepsilon)$ ($v_i = \gamma_{ij} v^j$) they are

$$D = \rho W \quad , \quad S_j = \rho h W^2 v_j \quad , \quad E = \rho h W^2 - p \quad (21)$$

Taking all the above relations together, the fundamental first-order, flux-conservative system reads

$$\frac{1}{\sqrt{-g}}\left(\frac{\partial\sqrt{\gamma}\mathbf{F}^0(\mathbf{w})}{\partial x^0} + \frac{\partial\sqrt{-g}\mathbf{F}^i(\mathbf{w})}{\partial x^i}\right) = \mathbf{s}(\mathbf{w}) \quad (22)$$

where the quantities $\mathbf{F}^\alpha(\mathbf{w})$ are

$$\mathbf{F}^0(\mathbf{w}) = (D, S_j, \tau) \quad (23)$$

$$\mathbf{F}^i(\mathbf{w}) = \left(D\left(v^i - \frac{\beta^i}{\alpha}\right),\right. \tag{24}$$

$$S_j\left(v^i - \frac{\beta^i}{\alpha}\right) + p\delta^i_j, \tag{25}$$

$$\left.\tau\left(v^i - \frac{\beta^i}{\alpha}\right) + pv^i\right) \tag{26}$$

and the corresponding sources $\mathbf{s}(\mathbf{w})$ are

$$\mathbf{s}(\mathbf{w}) = \left(0, T^{\mu\nu}\left(\frac{\partial g_{\nu j}}{\partial x^\mu} - \Gamma^\delta_{\nu\mu}g_{\delta j}\right),\right. \tag{27}$$

$$\left.\alpha\left(T^{\mu 0}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^0_{\nu\mu}\right)\right) \tag{28}$$

τ being $\tau \equiv E - D$, and $g \equiv \det(g_{\mu\nu})$ is such that $\sqrt{-g} = \alpha\sqrt{\gamma}$ ($\gamma \equiv \det(\gamma_{ij})$)

III.2.- The characteristic fields

Modern HRSC schemes use the characteristic structure of the hyperbolic system of conservation laws. In many Godunov-type schemes, the characteristic structure is used to compute either an exact or an approximate solution of a family of local Riemann problems at each cell interface. In characteristic based methods the characteristic structure is used to compute the local characteristic fields, which define the directions along which the characteristic variables propagate. In both these approaches, the characteristic decomposition of the Jacobian matrices of the nonlinear system of conservation laws is important, not only to compute the numerical fluxes at the interfaces, but because experience has shown that it facilitates a robust upgrading of the order of a numerical scheme.

The three 5×5-Jacobian matrices $\mathcal{B}^i$ associated to system (22) are

$$\mathcal{B}^i = \alpha\frac{\partial \mathbf{F}^i}{\partial \mathbf{F}^0}. \tag{29}$$

The full spectral decomposition of the above three 5×5-Jacobian matrices $\mathcal{B}^i$ can be found in Ibáñez *et al.* (2001).

For the sake of completeness let me include the expressions of the eigenvalues:

$$\lambda_0 = \alpha v^x - \beta^x \quad \text{(triple)} \tag{30}$$

which defines the *material waves*, and

$$\lambda_\pm = \frac{\alpha}{1 - v^2c_s^2}\left\{v^x(1 - c_s^2) \pm c_s\sqrt{(1 - v^2)[\gamma^{xx}(1 - v^2c_s^2) - v^xv^x(1 - c_s^2)]}\right\} - \beta^x \tag{31}$$

which are associated with the *acoustic waves.*

As the reader can easily notice the characteristic wave speeds, in the relativistic case, not only depend on the fluid velocity components in the wave propagation direction, but also on the normal velocity components. This coupling adds new numerical difficulties which are specific to relativistic fluid dynamics. Figures 1 and 2 will help the reader to have a clear idea about the behaviour of the characteristic fields (assuming that the vector of unknowns and fluxes depend only on the spatial coordinate x and time t). In both figures I have plotted the functions λ ($x-$direction) corresponding to a Schwarzschild spacetime, in terms of the modulus of velocity, for a given value of the local speed of sound ($c_s = 0.1$), and for three different values of the transversal velocity $v_t = \sqrt{(v^y)^2 + (v^z)^2}$: $v_t = 0, 0.4, 0.8$.

In Fig. 1, I have selected a particular value of the radial coordinate $r/2M = 10^5$, where M is the mass of the source, in order to mimic the asymptotically flat region. Hence the curves shown in this figure suggest the fundamental issues of the dynamics in a Minkowskian spacetime: i) The characteristic fields converge in the ultrarrelativistic regime (like the hypersonic regime in the classical case). ii) Unlike the Newtonian case, the influence of the the transversal velocity is very remarkable. For the same setup, in the classical case, the curves would be straight lines starting (in $v = 0$) at ± 0.1 and 0 for $\lambda_\pm$ and λ_0, respectively.

Fig. 2 is analogous to Fig. 1 but the value of $r/2M$ is 1.5. Hence, in this case I have emphasized the relativistic effects coming from the geometrical factors. The trend is clear, as far as the gravitational field becomes stronger: the dynamics looks like the hypersonic regime (the characteristic fields become parallel) and the characteristic velocities tend to zero in the limit when the lapse tends to zero.

We end this section pointing out that covariant formulations of the general relativistic hydrodynamic equations, alternative to the one described here, are available in the literature (Eulderink & Mellema, 1995; Papadopoulos & Font, 2000). These formulations are also suited for the used of advanced HRSC schemes. The corresponding characteristic structure can be found in the above references.

III.3.- Riemann Solvers in Relativistic Hydrodynamics

The scientific literature concerning with special relativistic Riemann Solvers (SRRS) has known a spectacular progress during the second half of 1990s. Although some of the SRRS proposed are a straigthforward extension of the corresponding in classical fluid dynamics, most of them have been specifically designed to handle the Riemann problem of the equations of (special) relativistic hydrodynamics (for perfect fluids). An up-to-dated list of the SRRS can be found in Martí & Müller (1999): i) Roe-type (Roe, 1981) SRRS (Martí, Ibáñez & Miralles, 1991). ii) HLLE (Harten et al, 1984) SRRS (Schneider et al., 1993). iii) The exact SRRS (Martí & Müller, 1994). iv) Two-Shock Approximation (Balsara, 1994). v) ENO (Essentially Non-Oscillatory, Shu & Osher, 1989) SRRS (Dolezal & Wong, 1995). vi) General relativistic extension of Roe RS (Eulderink & Mellema, 1995). vii) Upwind SRRS

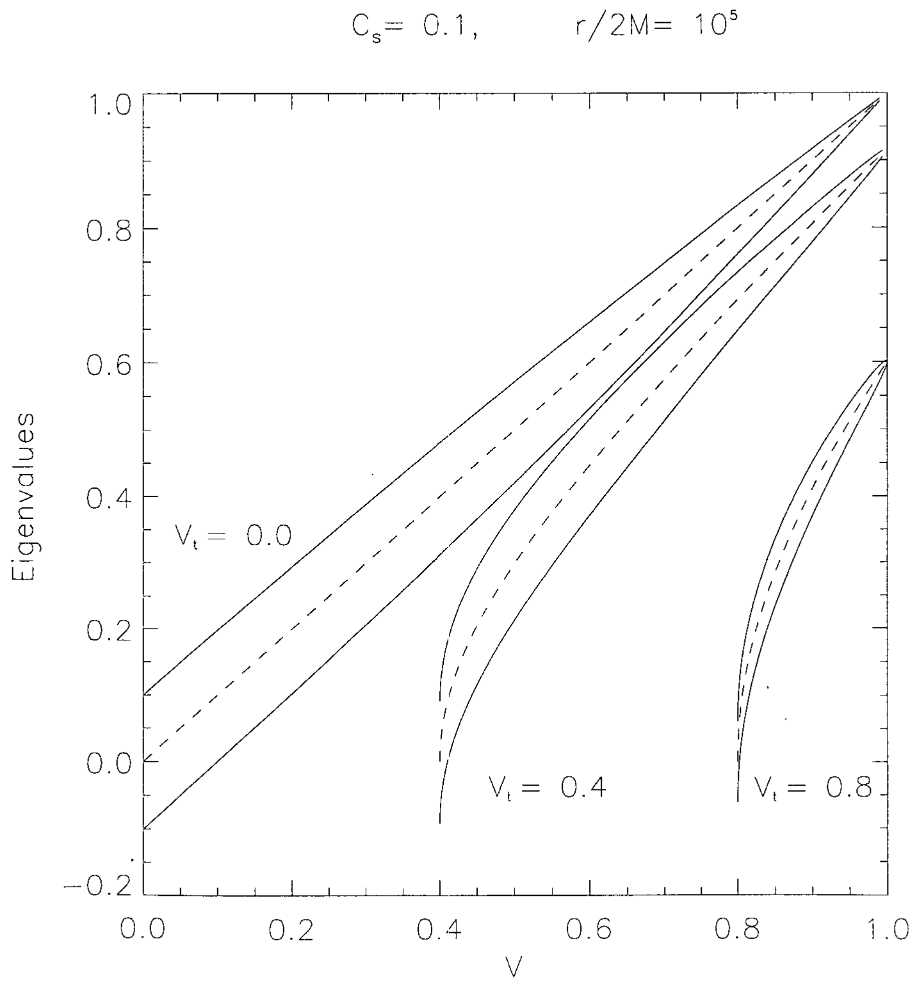

Fig. 1.— Eigenvalues ($x-$direction) as a function of the modulus of velocity. A Schwarzschild background has been considered with a value of the quantity $r/2M$ (M is the mass of the source) of 10^5 (i.e., an asymptotically Minkowskian spacetime). Three different values of the transversal velocity $v_t = \sqrt{(v^y)^2 + (v^z)^2}$ have been taken into account: $v_t = 0, 0.4, 0.8$. The local speed of sound is 0.1

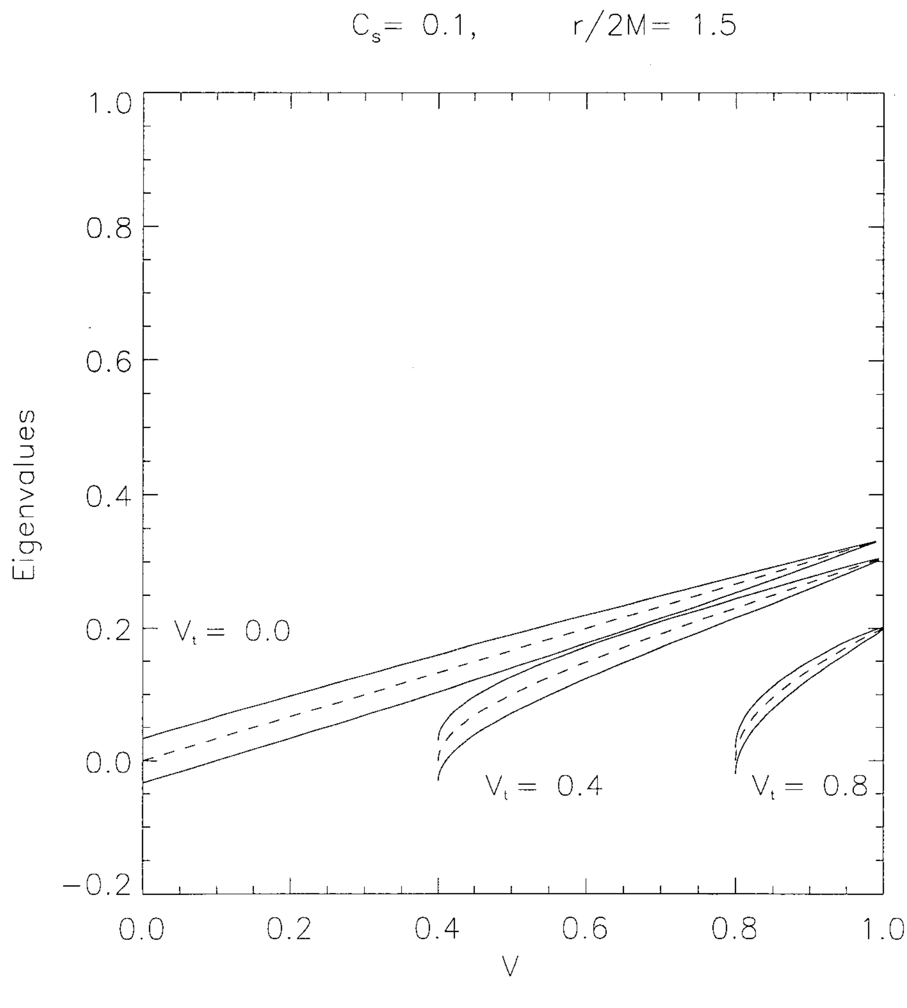

Fig. 2.— Eigenvalues (x−direction) as a function of the modulus of velocity. A Schwarzschild background has been considered with a value of the quantity $r/2M$ (M is the mass of the source) of 1.5 (i.e., a region where the gravitational field is strong). Three different values of the transversal velocity $v_t = \sqrt{(v^y)^2 + (v^z)^2}$ have been taken into account: $v_t = 0, 0.4, 0.8$. The local speed of sound is 0.1

(Falle & Komissarov, 1996). viii) Relativistic extension of PPM (Piecewise Parabolic Method, Martí & Müller, 1996). ix) Glimm SRRS (Wen, Panaitescu & Laguna, 1997). x) Iterative SRRS (Dai & Woodward, 1997). xi) Marquina's flux formula (Donat et al., 1998). xii) The exact SRRS for non-zero transversal velocities (Pons, Martí & Müller, 2000).

To end this first part, let me draw the reader's attention to the paper by Pons *et al.* (1998) in which we show how to extend any SRRS to the field of general-relativistic hydrodynamics.

IV. ASTROPHYSICAL APPLICATIONS

In this second part I am going to summarize the main results obtained by our group in the study of astrophysical systems where flows evolve reaching velocities near the speed of light and/or in the presence of strong gravitational fields (background or dynamical).

IV.1.- Relativistic Jets

In terms of the distance to the central object (a supermassive black hole) powering the nuclear activity in radio loud active galactic nuclei we can distinguish, in their associated relativistic jets, three main regions: *Subparsec scale*, *Parsec scale* and *Kiloparsec scale.*

At kiloparsec scales, the implications of relativistic flow speeds and/or relativistic internal energies in the morphology and dynamics of jets have been the subject of a detailed investigation: van Putten 1993, Martí, Müller & Ibáñez 1994, Duncan & Hughes 1994, Martí *et al.* 1995, Martí *et al.* 1997. Beams with large internal energies (*hot jets*) show little internal structure and relatively smooth cocoons allowing the terminal shock (the hot spot in the radio maps) to remain well-defined during the evolution. Their morphologies resemble those observed in naked quasar jets like 3C273 (Davis, Muxlow & Conway 1985). Highly supersonic models, in which kinematic relativistic effects dominate due to high beam Lorentz factors (*cold jets*), display extended overpressured cocoons. As noted by Martí *et al.* (1995), these overpressured cocoons can help to confine the jets during the early stages of evolution and even cause their deflection when propagating through non-homogeneous environments. The cocoon overpressure causes the formation of a series of oblique shocks within the beam in which the synchrotron emission is enhanced. In long term simulations (Scheck *et al.* 2001) the evolution is dominated by a strong deceleration phase during which large lobes of jet material, like the ones observed in many FRIIs (e.g., Cyg A, see Carilli *et al.* 1996), start to inflate around the jet's head. The numerical simulations reproduce some properties observed in powerful extragalactic radio jets (lobe inflation, hot spot advance speeds and pressures, deceleration of the beam flow along the jet) and can help to constrain the values of basic parameters (such as the particle density and the flow speed) in the jets of real sources.

The development of multidimensional relativistic hydrodynamic codes has allowed the simulation of parsec scale jets and superluminal radio components for the first time. The presence of emitting flows at almost the speed of light enhance the importance of relativistic effects in the appearance of these sources (relativistic Doppler boosting, light aberration, time delays). Hence, models should use a combination of hydrodynamics and synchrotron radiation transfer to compare them with observations. In these models, moving radio components are obtained from perturbations in steady relativistic jets. These jets propagate through pressure decreasing atmospheres causing them to expand and accounting for the observed jet opening angles. Where pressure mismatches exist between the jet and the surrounding atmosphere reconfinement shocks are produced. The energy density enhancement produced downstream from these shocks can give rise to stationary radio knots like those observed in many VLBI sources. Superluminal components are produced by triggering small perturbations in these steady jets which propagate at almost the jet flow speed. The first radio emission simulations from high-resolution three-dimensional relativistic hydrodynamic jets have been presented in Aloy *et al.* (1999b, 2000a). They have been generated running GENESIS (Aloy *et al.* 1999a), an optimized and parallelized 3D special relativistic hydro-code, which is suited for massively parallel computers with distributed memory. A general-relativistic version of GENESIS is currently in progress.

IV.2.- Jets from Collapsars

Various catastrophic collapse events have been proposed to explain the energies released in a GRB including mergers of compact binaries (Pacyński 1986; Goodman 1986; Eichler *et al.* 1989; Mochkovitch *et al.* 1993), collapsars (Woosley 1993) and hypernovae (Pacyński 1998). According to the current view these models require a common engine, namely a stellar mass black hole (BH) which accretes up to several solar masses of matter. A fraction of the gravitational binding energy released by accretion is thought to power a pair fireball. If the baryon load of the fireball is not too large, the baryons are accelerated together with the $e^+ e^-$ pairs to Lorentz factors $> 10^2$ (Cavallo & Rees 1978). Such relativistic flows are supported by radio observations of GRB 980425 (Kulkarni *et al.* 1998).

MacFadyen & Woosley (1999, MW99) have explored the evolution of rotating helium stars ($M_\alpha \gtrsim 10\,M_\odot$) whose iron core collapse does not produce a successful outgoing shock, but instead forms a BH surrounded by a compact accretion torus. Assuming that the efficiency of energy deposition is higher in the polar regions, MW99 obtain relativistic jets along the rotation axis, which remain highly focused and seem capable of penetrating the star. However, as their simulations were performed with a Newtonian code, they obtain speeds in the jet flow which are appreciably superluminal. Using a collapsar progenitor, provided by MacFadyen & Woosley, we have simulated (Aloy *et al.*, 2000b) the propagation of an axisymmetric jet through a collapsing rotating massive star with the multidimensional relativistic hydrodynamic code GENESIS. The jet forms as a consequece of an assumed energy deposition in the range $10^{50} - 10^{51}$ ergs s^{-1} within a 30^0 cone around the rotation axis.

We have assumed a spacetime corresponding to a Schwarzschild BH. Effects due to the self-gravity of the star on the dynamics are neglected. The equation of state includes the contributions of non-relativistic nucleons treated as a mixture of Boltzmann gases, radiation, and an approximate correction due to e^+e^--pairs. Complete ionization is assumed, and the effects due to degeneracy are neglected. We advect nine non-reacting nuclear species which are present in the initial model: C^{12}, O^{16}, Ne^{20}, Mg^{24}, Si^{28}, Ni^{56}, He^4, neutrons and protons.

The jet flow is strongly beamed, spatially inhomogeneous, and time dependent. The jet reaches the surface of the stellar progenitor intact. At breakout, the maximum Lorentz factor of the jet flow is 33. Ábfter breakout, the jet accelerates into the circumstellar medium, whose density is assumed to decrease exponentially and then become constant ($\approx 10^{-5}$ g cm^{-3}. Outside the star, the flow begins to expand laterally but the beam remains very well collimated. When the simulation ends, the Lorentz factor has increased to 44.

IV.3.- Relativistic Bondi-Hoyle Accretion

Recent discoveries made in the field of X-ray Astronomy have greatly increased interest in the physics of accretion flows around compact objects (neutron stars and black holes). Analysis of quasi-periodic oscillations (QPOs), in the kHz range, observed in neutron star X-ray binaries (van der Klis 1997) may lead to measurements of the precession of the accretion disk, due to the Lense-Thirring effect (Stella & Vietri 1998). Same line of reasoning applied to QPOs observed in the black hole candidate GRS 1915+105 (Morgan, Remillard & Greiner 1997) suggests that GRO J1655-40 and GRS 1915+105 are spinning at a rate close to the maximum theoretical limit (Cui, Zhang & Chen 1998). The iron Kα emission line in the active galaxy MCG-6-30-15 (Bromley, Miller, & Pariev 1998) furnishes further evidence that (rotating) black holes are at the center of active galactic nuclei.

Historically, the canonical astrophysical scenario in which matter is accreted in a non-spherical way by a compact object was suggested by Hoyle & Lyttleton (1939) and Bondi & Hoyle (1944). This will be referred to as the *Bondi-Hoyle-Lyttleton* accretion onto a Schwarzschild black hole. Using Newtonian gravity these authors studied the accretion onto a gravitating point mass moving with constant velocity through a nonrelativistic gas which is at rest and has a uniform density at infinity. Since then, this pioneering analytic work has been numerically investigated, for a finite size accretor, by a great number of authors over the years (see, e.g., Font & Ibáñez 1998a, for an up-to-date reference list).

In a series of papers (Font & Ibáñez 1998a, 1998b; Font, Ibáñez & Papadopoulos 1998, 1999) the authors have made an extensive numerical study of the relativistic extension of the Bondi-Hoyle-Lyttleton scenario. In particular, in Font, Ibáñez & Papadopoulos (1998, 1999) a detailed analysis is made of the morphology and dynamics of the flow evolving in the equatorial plane of a Kerr black hole. The analysis made is novel in its use of *the Kerr-Schild (KS) coordinate system*, which is the simplest within the family of *horizon adapted coordinate systems*, introduced in Papadopoulos & Font(1999) where all fields, i.e., metric, fluid and electromagnetic fields are free of coordinate singularities at the event horizon. This procedure allows to perform accurate numerical studies of relativistic accretion flows around black holes since it is possible to extend the computational grid inside the black hole horizon. A HRSC technique (which makes use of a linearized Riemann solver) has been used to solve system (22) in Boyer-Lindquist (BL) and also KS coordinates. In BL coordinates, for a near-extreme Kerr black hole, the shock wraps many times before reaching the horizon due to coordinate effects. This is a pathology of the BL system associated to the collapse of the lapse function at the horizon. The wrapping in the shock wave has an important and immediate consequence: its computation in BL coordinates, although possible in principle, would be much more challenging than in KS coordinates, and the numerical difficulties would increase the closer to the horizon one would impose the boundary conditions in the BL framework.

Finally, in Brandt *et al.* (2000) we make some studies of the spherical and axisymmetric accretion onto a dynamic black hole, the fully dynamical evolution of imploding shells of dust with a black hole, the evolution of matter in rotating spacetimes, the gravitational radiation induced by the presence of the matter fields and the behavior of apparent horizons through the evolution.

IV.4.- General Relativistic Stellar Core Collapse

In the case of spherically symmetric spacetimes the general relativistic equations can be given in a simple way which resembles the Newtonian hydrodynamics. To this aim the choice of coordinates is crucial. The use of Schwarzschild-type coordinates (Bondi 1964) leads to a simple general relativistic extension of the Eulerian Newtonian hydrodynamics. In terms of slicing of space-time, Schwarzschild-type coordinates are the realization of a *polar time slicing* and *radial gauge* (see Gourgoulhon 1991).

We have studied (Romero *et al.* 1996) the general-relativistic spherically symmetric stellar core collapse, paying particular attention to the numerical treatment of the formation and propagation of strong shocks (in the framework of the so-called prompt mechanism of type II Supernova) extending HRSC techniques to the general-relativistic hydrodynamic equations. Details on the particular equations to be solved can be found in the above reference. A very simple way of modelling the essential features of the stellar core collapse of massive stars is to incorporate a simple equation of state like that of an ideal gas, but taking an adiabatic exponent, Γ, which depends on the density according to:

$$\Gamma = \Gamma_{min} + \eta(\log \rho - \log \rho_b) \quad , \tag{32}$$

with $\eta = 0$ if $\rho < \rho_b$ and $\eta > 0$ otherwise (van Riper 1979). In Romero *et al.* (1996) we have considered the collapse of a white dwarf-like configuration, with a gravitational mass of 1.3862 $M_\odot$, and two sets of values for the parameters Γ_{min}, η and ρ_b: $\{1.33,\ 1,\ 2.5\times10^{14}$ gcm^{-3} $\}$ (*model A*) and $\{1.33,\ 5,\ 2.5\times10^{15}$ gcm^{-3} $\}$ (*model B*).

The following main results obtained by Romero *et al.* (1996) merit to be pointed out: i) The formation and evolution of a shock is sharply solved in one or two zones and is free of spurious oscillations. ii) The conservative features of the code, consistent with the conservation laws of baryonic mass and gravitational mass (or binding energy).

Let me draw the reader's attention towards a recent paper by Dimmelmeier, Font & Müller (2001) where, for first time, the authors make use of HRSC techniques for evolving matter in a dynamical spacetime described assuming the so-called conformal flatness condition. The authors make an exhaustive analysis of the gravitational waveforms generated during axisymmetric relativistic rotational core collapse. This work is an important step towards further studies of fully multidimensional general-relativistic stellar core collapse.

Astrophysical applications using the *characteristic formulation* of general relativity and hydrodynamics (Papadopoulos & Font, 2000) in investigations of collapse of supermassive stars and gravitational waves from accreting black holes can be found in Linke *et al.* (2001) and Papadopoulos & Font (2001), respectively.

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