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A NOTE ON THE VARIATIONAL FORMULA ON TEICHMÜLLER SPACE

Ki Sung Park

Abstract. In this paper, we study the first and second formulas of energy functions on the Teichmüller spaces and prove the relation between index and nullity of Jacobi operator.

1. Introduction

Let (M, g) and (N, h) be two compact Riemannian manifolds and $\phi : M \to N$ be a smooth map. Harmonic maps are extremals of the energy functional

$$E(\phi) = \int_M e(\phi)\, v_g,$$

where $e(\phi) = \frac{1}{2}|d\phi|^2$ is the energy density and v_g is the canonical volume element.

The map ϕ is harmonic if and only if it satisfies the Euler-Lagrange equation

$$\tau(\phi) = \operatorname{div}(d\phi) = 0.$$

The field $\tau(\phi)$ is called the **tension field of** ϕ; it is a section of $\phi^{-1}T(N)$. The basic existence problem for harmonic maps concerns the minima and critical point of energy homotopy classes of maps from M to N, when there is a harmonic map in the homotopy class $\mathcal{H}$. This problem has been studied extensively and the answers depend on the manifolds and the homotopy class. To obtain existence of solution in all dimensions without conditions on the manifolds, Eells-Lamaire [6] considered another problems of calculus of variations. They defined the exponential energy of ϕ as $E_e(\phi) = \int_M \exp(\frac{1}{2}|d\phi|^2)v_g$, and say that a smooth extremal of E_e is an exponential harmonic maps.

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Schoen and Yau [4] defined an energy function E_ϕ associated with harmonic maps on the Teichmüller spaces $T(R)$ for fixed Riemann surface of genus γ and using the energy function, showed the existence of incompressible minimal surfaces.

In this paper, we study the first and second formulas of energy functions on the Teichmüller space $T(R)$ and prove the following statement: If N is negative curved, then

$$\text{index } A + \text{nullity } A \leq 2 \dim_{\mathbb{C}} K(D^*).$$

2. Main theorems

Let N be an n dimensional compact Riemannian manifold and M be a compact Riemannian surface of genus γ. Let ϕ be a smooth map from M to N. Then ϕ induces the map $\phi_\#$ of $\pi_1(M,*)$ into $\pi_1(N,\phi(*))$, where $*$ is a fixed point of M. Let $L_1^2(M,N)$ denote the space of maps of M into N having square integrable first derivations in the distribution sense. Schoen and Yau [4] proved that there exists an energy minimizing harmonic map among $\{\, f \in L_1^2(M,N) \mid f_\# = \tau^{-1}\phi_\#\tau \,\}$, where τ is some curve from $f(*)$ to $\phi(*)$. Let R be a fixed compact Riemannian surface of genus γ. A pair of a compact Riemannian surface M of genus γ and a differential homeomorphism f of R onto M is denoted by (M,f). We define an equivalence relation for pairs as follows. (M_2,f_2) is said to be equivalent to (M_1,f_1) if and only if there exists a biholomorphic map h of M_1 onto M_2 such that h is homotopic to $f_2f_1^{-1}$. The space of all equivalence classes is called the **Teichmüller space $T(R)$ of R.**

Let $[(M,f)]$ denote the points of $T(R)$ and ϕ be a smooth map of R into an n dimensional Riemannian manifold N. Let $\tilde{f}_1$ be the energy minimizing harmonic map of M_1 into N for ϕf_1^{-1}. When (M_2,f_2) is equivalent to (M_1,f_1) by a biholomorphic map h, if $\tilde{f}_2$ is an energy minimizing harmonic map of M_2 into N for ϕf_2^{-1}, $\tilde{f}_2h$ becomes an energy minimizing harmonic map of M_1 into N for ϕf_1^{-1}. Thus the energy of $\tilde{f}_1$ and $\tilde{f}_2$ are the same, which defines the energy function E_ϕ on $T(R)$ by giving the energy of a correspondent energy minimizing harmonic map at $[(M,f)]$. Let ζ be a parameter of a neighborhood of a point $[(M,f)] \in T(R)$. Then there exists a Riemannian metric g_ζ

on R whose scalar curvature is -1, which gives the complex structure corresponding to ζ. Let $[g_\zeta]$ denote the point of $T(R)$ for ζ. Furthermore, for the almost complex structure J_ζ corresponding to $[g_\zeta]$, we may consider that $[J_\zeta]$ also denotes the same point of $T(R)$. We denote by (R, g_ζ) the compact Riemannian surface compatible with g_ζ. We may use R as a compact Riemannian surface (R, g_ζ). Let g_t be a one parameter family in g_ζ and $S(g_t) = S_t$ the harmonic map for ϕ with respect to g_t. Then the energy function along t is defined by the energy $E(g_t, S_t)$ of S_t with respect to g_t:

$$E_\phi([g_t]) = E(g_t, S_t) = \int_R g_t^{ij} \langle \frac{\partial S_t}{\partial x^i}, \frac{\partial S_t}{\partial x^j} \rangle \sqrt{g_t}\, dx^1 dx^2$$

where $\sqrt{g_t}$ means $\sqrt{\det(g_{t\,ij})}$ and (x^1, x^2) is a local coordinate system of R. The harmonic map equation is given by

$$\frac{1}{\sqrt{g_t}} \frac{\partial}{\partial x^i} (g_t^{ij} \sqrt{g_t} \frac{\partial S_t^\alpha}{\partial x^j}) + \Gamma^\alpha_{\gamma\beta}(S_t) \frac{\partial S_t^\gamma}{\partial x^i} \frac{\partial S_t^\beta}{\partial x^j} g_t^{ij} = 0.$$

Let $h_{t_{ij}}$ be the first variation of g_t. Then $h_{t_{ij}}$ is trace free for $g_{t_{ij}}$, because g_t has a constant scalar curvature -1 and we may consider that h_{ij} is divergence free for $g_{t_{ij}}$ which implies that $h_{zz}dz^2$ is a holomorphic quadratic differential for (R, g). The first and second differential of g_t^{ij} are given by

$$(g_t^{ij})' = -g_t^{i\ell} g_t^{jm} (h_{t\ell m})',$$
$$(g_t^{ij})''_{t=0} = h^{i\ell} g^{jm} h_{\ell m} + g^{i\ell} h^{jm} h_{\ell m} - g^{i\ell} g^{jm} (g_{t\ell m})''_{t=0}.$$

Since

$$(\sqrt{g_t})' = \frac{1}{2} g_t^{ij} (g_{t_{ij}})' \sqrt{g_t},$$

and $h_{t_{ij}}$ is trace free for g_{ij}, we get

$$-h^{ij} h_{ij} + g^{ij} (g_{t_{ij}})''_{t=0} = 0.$$

Let $V(h)$ be the variation vector field $(S_t)'_{t=0}$. Then the first variation of $E_\phi([g_t])$ is given by

$$\begin{aligned} DE_\phi([g_t])h_t = &\int_R (g_t^{ij})' \langle \frac{\partial S_t}{\partial x^i}, \frac{\partial S_t}{\partial x^j} \rangle \sqrt{g_t}\, dx^1 dx^2 \\ &- \int_R \langle g_t^{ij} \sqrt{g_t} \frac{\partial S_t}{\partial x^i}, (\frac{\partial S_t}{\partial x^j})' \rangle\, dx^1 dx^2. \end{aligned}$$

Since S_t is a harmonic map with respect to g_t,

$$DE_\phi([g_t])h_t = \int_R (g_t^{ij})' \langle \frac{\partial S_t}{\partial x^i}, \frac{\partial S_t}{\partial x^j} \rangle \sqrt{g_t}\, dx^1 dx^2$$

holds. Thus the second variation $E_\phi([g_t])$ at $t=0$ becomes

$$D^2E_\phi([g])(h,h) = \int_R [\, (g_t^{ij})''_{t=0} \langle \frac{\partial S}{\partial x^i}, \frac{\partial S}{\partial x^j} \rangle \sqrt{g}$$
$$+ 2(g_t^{ij})'_{t=0} \langle (\frac{\partial S_t}{\partial x^i})'_{t=0}, \frac{\partial S}{\partial x^j} \rangle \sqrt{g}\,]\, dx^1 dx^2.$$

PROPOSITION 1. [3] *Let $[g_t]$ be a critical point of E_ϕ. Then*

$$D^2E_\phi([g])(h,h) = \int_R [\, \mu h^{ij} h_{ij} - 2h^{ij} \langle \frac{\partial V(h)}{\partial x^i}, \frac{\partial S}{\partial x^j} \rangle \sqrt{g}\,]\, dx^1 dx^2,$$

where $\mu((dx^1)^2 + (dx^2)^2)$ is the induced metric on each coordinate neighborhood.

Let D be the Fredholm operator acting on the space of sections of $S_*T^{(0,1)}R$ and transfering the sections to the sections of $S_*T^{(0,1)}R \otimes T^{*(1,0)}R$ such that

$$D : \xi \to \nabla_{\frac{\partial}{\partial z}} \xi dz.$$

Let D^* be the adjoint operator of D. Then the kernel $K(D^*)$ of D^* is the space of holomorphic sections of $S_*T^{(0,1)}R \otimes T^{*(1,0)}R$. We can define a map of $K(D^*)$ into Q by

$$\xi_z dz \to \langle \xi_z, \frac{\partial S}{\partial Z} \rangle dz^2.$$

THEOREM 2. *This map is an injective linear map. S is an immersion if and only if it is injective.*

Proof. Since

$$c_1(S_*T^{(0,1)}R \otimes T^{*(1,0)}R) = -b + 4(\gamma - 1),$$

where b is the number of branch points,

$$\dim_c K(D^*) = \begin{cases} 3\gamma - 3 - 2b & \text{if } b \le 2\gamma - 4 \\ 2\gamma - 1 - [-\frac{b}{2}] & \text{if } 2\gamma - 3 \le b \le 4\gamma - 4 \\ 0 & \text{if } b \ge 4\gamma - 3 \end{cases}.$$

Here $\dim Q = 3\gamma - 3 \ge \dim K(D^*)$. □

Let Z be the space $\{\, q_{zz} dz^2 \in Q \mid \frac{1}{\lambda} q_{zz} \frac{\partial S}{\partial \bar{z}} dz$ is orthogonal to $K(D^*) \,\}$.

Lemma 3. *Let s be a section of S^*TN. Then there exists $h_{zz}dz^2 \in Q$ such that*

$$\left(\frac{1}{\lambda}h_{zz} + \frac{1}{\mu}\sigma_{zz}^{S^\perp}\right)\frac{\partial S}{\partial \bar{z}}dz$$

is orthogonal to $K(D^*)$. In particular, $h_{zz}dz^2$ is determined up to the element of Z.

Lemma 4. *Let s be a Jacobi field. Let $h_{zz}dz^2$ be an element obtained in Lemma 3. Then we get W such that*

$$DW^{T(0,1)} = \left[\frac{1}{\lambda}h_{zz} + \frac{2}{\mu}\sigma_{zz}^{S}\right]\frac{\partial S}{\partial \bar{z}}dz.$$

In particular, W is determined up to the elements η such that

$$D\eta^{T(0,1)} = \frac{1}{\lambda}q_{zz}\frac{\partial S}{\partial \bar{z}}dz,$$

where $q_{zz}dz^2 \in Z$.

Proof. By the alternative theorem for Fredholm operators, we get a solution. □

Eells-Lemaire [6] proved that, for a harmonic map of (R, g_ζ) into N whose second variation for the harmonic map is non-degenerate, there exists a harmonic map $S(g_\zeta)$ of (R, g_ζ) into N with smooth dependence on S. If N has negative curved, then it is well known that the second variation for non-constant harmonic map is positive.

Let index A and nullity A denote the index and the nullity of the Jacobi operator, respectively. Then, by Lemma 3 and Lemma 4, we get the following theorem.

Theorem 5. *If N is negative curved, then*

$$\text{nullity of } E_\phi = \text{nullity } A + 6\gamma - 6 - 2\dim_c K(D^*).$$

Since nullity of E_ϕ + index of $E_\phi \leq 6\gamma - 6$, we have the following statement.

Corollary 6. *If N is negative curved, then*

$$\text{index } A + \text{nullity } A \leq 2\dim_c K(D^*).$$

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Ki Sung Park
Department of Mathematics
Kangnam University
Yongin 449–702, Korea