

CHARACTERIZATIONS OF INFINITELY DIVISIBLE AND TYPE G PROCESSES

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ABSTRACT. We generalize LePage-type representation of infinitely divisible processes and survey type G processes. Finally, we get integral representation and some inequality of processes of type G.

1. Introduction

Let $\{\xi_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. random vectors related to some measure and let Γ_j be the j -th arrival time of unit rate Poisson process. Consider the series representation

$$\sum_{j=1}^{\infty} R(\Gamma_j, \xi_j) f(\xi_j),$$

where, R is a function related to control measure and Borel measurable function and f is a function belonging to Musielak-Orlicz space. Series representations involving arrival times in a Poisson process have been given by Ferguson and Klass [1], for real independent increment processes without Gaussian components and with positive jumps. LePage series representation is developed for infinitely divisible processes by Rajput and Rosinski ([3],[4],[5],[7]) and tail behavior of subadditive functionals of paths of infinitely divisible processes is obtained ([6]).

Chapter 2 is to review some basic definitions and some example. In chapter 3, we generalize LePage-type representation of infinitely divisible processes. In chapter 4, we survey a large class of infinitely divisible processes, known as, type G processes and get integral representation and some inequality of processes of type G.

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2. Preliminaries

We denote, by S , an arbitrary non-empty set and, by \mathcal{S} , a δ -ring of subsets of S . Let $\Lambda = \{\Lambda(A) : A \in \mathcal{S}\}$ be a real stochastic process defined on some probability space (Ω, \mathcal{F}, P) . We begin with some definitions.

DEFINITION 2.1. *A random measure Λ is said to be an independently scattered random measure if for every sequence $\{A_n\}$ of disjoint sets in \mathcal{S} , the random variables $\Lambda(A_n)$, $n = 1, 2, \dots$ are independent and if $\cup_n A_n$ belong to \mathcal{S} then we also have $\Lambda(\cup_n A_n) = \sum_n \Lambda(A_n)$ a.s, where the series is assumed to converge almost surely.*

DEFINITION 2.2. *A random measure Λ is said to be an independently scattered infinitely divisible random measure if for each $A \in \mathcal{S}$, $\Lambda(A)$ is infinitely divisible random variable and Λ is independently scattered random measure.*

We know that characteristic function of infinitely divisible random variable $\Lambda(A)$ can be written in the Khintchin-Lévy form;

$$Ee^{iu\Lambda(A)} = \exp\left\{iu\nu_0(A) - \frac{1}{2}u^2\nu_1(A) + \int_R e^{iux} - 1 - iuxI_{|x| \leq \tau}(x)F_A(dx)\right\} \quad (2.1)$$

for some $\tau > 0$, where $-\infty < \nu_0(A) < \infty$, $0 \leq \nu_1(A) < \infty$ and F_A is a Lévy measure on R .

DEFINITION 2.3. *A σ -finite measure ν on $\sigma(\mathcal{S})$ is said to be a control measure of the random measure Λ if Λ and ν have the same families of zero sets.*

EXAMPLE 2.1. *Let ν_0, ν_1 and F be as in (2.1) and define*

$$\nu(A) := |\nu_0|(A) + \nu_1(A) + \int_R \min\{1, x^2\}F_A(dx), \quad A \in \mathcal{S},$$

Then $\nu : \sigma(\mathcal{S}) \rightarrow [0, \infty]$ is a control measure of Λ , where $\sigma(\mathcal{S})$ is the smallest σ -field generated by \mathcal{S} .

From [2], we adopt the following definition of a process of type G .

DEFINITION 2.4. *A process $\{X_t, t \in T\}$ is of type G if there is a function $\psi : [0, \infty) \rightarrow R$ with completely monotone derivative on $(0, \infty)$,*

$\psi(0) = 0$, and a σ -finite measure λ on the cylindrical σ -field of R^T such that

$$E \exp \sum_{j=1}^n \theta_j X_{t_j} = \exp \left\{ - \int_{R^T} \psi(2^{-1} | \sum_{j=1}^n \theta_j s(t_j) |^2) \lambda(ds) \right\}. \quad (2.2)$$

3. Series representation

Let λ be an arbitrary but fixed control measure of Λ .

LEMMA 3.1. *Let F be as in (2.1). Then there exists a unique σ -finite measure F on $\sigma(\mathcal{S}) \times \mathcal{B}(R)$ such that*

$$F(A \times B) = F_A(B), \quad \text{for all } A \in \mathcal{S}, B \in \mathcal{B}(R). \quad (3.1)$$

Moreover, there exists a function $Q : S \times \mathcal{B}(R) \rightarrow [0, \infty]$ such that

- (i) $Q(s, \cdot)$ is a Borel measure on $\mathcal{B}(R)$ for every $s \in S$,
- (ii) $Q(\cdot, B)$ is a Borel measurable function, for every $B \in \mathcal{B}(R)$,
- (iii) $\int_{S \times R} h(s, x) F(ds, dx) = \int_S [\int_R h(s, x) Q(s, dx)] \lambda(ds)$ for every $\sigma(\mathcal{S}) \times \mathcal{B}(R)$ -measurable function $h : S \times R \rightarrow [0, \infty]$.
- (iv) $\int_R (1 \wedge x^2) Q(s, dx) < \infty$, for every $s \in S$,
- (v) $\Phi_{\Lambda(A)}(u) = \exp \{ \int_A K(u, s) \lambda(ds) \}$, where,

$$K(u, s) = iua(s) - \frac{1}{2}u^2\sigma^2(s) + \int_R (e^{iux} - 1 - iuxI_{\{|x| \leq \tau\}}(x)) Q(s, dx),$$

$$a(s) = \frac{d\nu_0}{d\lambda}(s), \quad \sigma^2(s) = \frac{d\nu_1}{d\lambda}(s).$$

- (vi) $\lambda\{s \in S : a(s) = \sigma^2(s) = Q(s, R) = 0\} = 0$.

Proof. By [3, Lemma 2.3], there exists a unique σ -finite measure F on $\sigma(\mathcal{S}) \times \mathcal{B}(R)$ satisfying (3.1) and we can find a function $\rho : S \times \mathcal{B}(R) \rightarrow [0, \infty]$ such that $\rho(s, \cdot)$ is a Lévy measure on $\mathcal{B}(R)$ for every $s \in S$, $\rho(\cdot, B)$ is a Borel measurable function for every $B \in \mathcal{B}(R)$ and

$$\int_{S \times R} h(s, x) F(ds, dx) = \int_S \int_R h(s, x) \rho(s, dx) \lambda(ds)$$

for every $\sigma(\mathcal{S}) \times \mathcal{B}(R)$ -measurable function $h : S \times R \rightarrow [0, \infty]$.

Since λ and ν are equivalent σ -finite measures on $\sigma(\mathcal{S})$, there exists a strictly positive and finite version ψ of the Radon-Nikodym derivative $d\nu/d\lambda$. Put

$$Q(s, dx) = \psi(s)\rho(s, dx).$$

Then (i), (ii) and (iii) and (v) follow because

$$F_A(B) = F(A \times B) = \int_A \int_R I_B(x)Q(s, dx)\lambda(ds).$$

Since $\rho(s, \cdot)$ is a Lévy measure, (iv) is satisfied. Finally, note that $A_0 = \{s : a(s) = \sigma^2(s) = Q(s, R) = 0\}$ is a Λ -zero set, so that $\lambda(A_0) = 0$. \square

Let q be a non-negative number such that

$$E|\Lambda(A)|^q < \infty \quad \text{for all } A \in \mathcal{S}.$$

Define

$$U(u, s) = ua(s) + \int_R (uxI_{\{|ux| \leq 1\}}(x) - uxI_{\{|x| \leq 1\}}(x))Q(s, dx)$$

and for $0 \leq p \leq q$, $u \in R$ and $s \in \mathcal{S}$,

$$\Phi_p(u, s) = U^*(u, s) + u^2\sigma^2(s) + V_p(u, s),$$

where

$$U^*(u, s) = \sup_{|c| \leq 1} |U(cu, s)|,$$

$$V_p(u, s) = \int_{-\infty}^{\infty} \{|ux|^p I_{\{|ux| > 1\}} + |ux|^2 I_{\{|ux| \leq 1\}}\}Q(s, dx).$$

We define the so-called Musielak-Orlicz space

$$L_{\Phi_p}(\mathcal{S}, \lambda) = \{f \in L_0(\mathcal{S}, \lambda) : \int_S \Phi_p(|f(s)|, s)\lambda(ds) < \infty\}.$$

The space $L_{\Phi_p}(\mathcal{S}, \lambda)$ is complete linear metric space with norm defined by

$$\|f\|_{\Phi_p} = \inf\{c > 0 : \int_S \Phi_p(c^{-1}|f(s)|, s)\lambda(ds) \leq 1\}.$$

Let $\lambda^{(1)}$ be an arbitrary probability measure on $(S, \sigma(\mathcal{S}))$ equivalent to λ . Set

$$R(r, s) := I\left(r \frac{d\lambda^{(1)}}{d\lambda}(s), s\right), \quad r > 0, \quad s \in S,$$

where,

$$I(r, s) = \inf\{x > 0 : Q(s, [-x, x]^c) \leq r\}, r > 0.$$

Define

$$\mathcal{F}_f(A) = \int_0^\infty \int_S I_{A \setminus \{0\}}(R(r, s)f(s))\lambda^{(1)}(ds)dr.$$

THEOREM 3.1. *Suppose $f \in L_{\Phi_0}$. Then $\mathcal{F}_f(A)$ is a Lévy measure.*

Proof. Since, for every $x \geq 0$ and $s \in S$,

$$\text{Leb}\{r > 0 : R(r, s) > x\} = \frac{d\lambda}{d\lambda^{(1)}}(s)Q(s, [-x, x]^c),$$

we know

$$\mathcal{F}_f(A) = \int_S \int_0^\infty I_{A \setminus \{0\}}(xf(s))Q(s, dx)\lambda(ds).$$

By [3, Proposition 2.6],

$$\exp\{iua_{f,\tau} + \int_R (e^{iux} - 1 - iuxI_{\{|x| \leq \tau\}})\mathcal{F}_f(dx)\}$$

is a characteristic function of $\int_S f(s)\Lambda(ds)$, where,

$$\begin{aligned} a_{f,\tau} &= \int_S f(s)a(s)\lambda(ds) + \int_S \int_R (f(s)xI_{\{|f(s)x| \leq \tau\}} - \\ &\quad f(s)xI_{\{|x| \leq \tau\}})Q(s, dx)\lambda(ds), \end{aligned}$$

and we get

$$\begin{aligned} \int_{\{|x| \leq 1\}} x^2 \mathcal{F}_f(ds) &= \int_S \int_{\{|f(s)x| \leq 1\}} |f(s)x|^2 Q(sdx)\lambda(ds) \\ &\leq \int_S |\Phi_0(|f(s)|, s)\lambda(ds) < \infty, \end{aligned}$$

$$\begin{aligned} \int_{\{|x| > 1\}} \mathcal{F}_f(ds) &= \int_S \int_{\{|f(s)x| > 1\}} Q(sdx)\lambda(ds) \\ &\leq \int_S |\Phi_0(|f(s)|, s)\lambda(ds) < \infty. \end{aligned}$$

Therefore, we conclude that \mathcal{F}_f is a Lévy measure. \square

Let $\{\xi_j\}_{j=1}^\infty$ be a sequence of i.i.d. random vectors taking values in (S, \mathcal{S}) with $\mathcal{L}(\xi_j) = \lambda^{(1)}$ and let Γ_j be the j -th arrival time of unit rate Poisson process. Then the following theorem holds.

THEOREM 3.2. Suppose that $f \in L_{\Phi_p}$ for some $p \geq 1$. Then

$$M_n(f) := \sum_{j=1}^n R(\Gamma_j, \xi_j) f(\xi_j) - C_f(\Gamma_n)$$

converges to $M_\infty(f)$ a.s. and in L^p as $n \rightarrow \infty$, where,

$$C_f(t) = \int_0^t \int_S R(r, s) f(s) \lambda^{(1)}(ds) dr.$$

Proof. We know that \mathcal{F}_f is a Lévy measure by Theorem 3.1. Since

$$\begin{aligned} \int_{\{|x|>1\}} |x|^p \mathcal{F}_f(dx) &= \int_S \int_{\{|f(s)x|>1\}} |xf(s)|^p Q(s, dx) \lambda(ds) \\ &\leq \int_S \Phi_p(|f(s)|, s) \lambda(ds) < \infty. \end{aligned}$$

Therefore, it follows from [5, Theorem 3.1]. \square

COROLLARY 3.1. The characteristic function of $M_\infty(f)$ is

$$\exp\{iua_f + \int_R (e^{iux} - 1 - iux) \mathcal{F}_f(dx)\},$$

where, $a_f = \int f(s)a(s)\lambda(ds)$.

4. Type G Process

We consider a large class of symmetric infinitely divisible processes, known as processes of type G, whose marginal distributions are variance mixtures of the normal distribution. The function ψ in definition 2.4 can be written as

$$\psi(y) = \int_R (1 - e^{-yt^2}) \sigma(dt), \quad y \geq 0,$$

where σ is a symmetric Lévy measure on R (i.e. satisfies $\int_R (1 \wedge t^2) \sigma(dt) < \infty$).

LEMMA 4.1. ([2]) Define

$$\rho(B) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma(u^{-1}B) e^{-\frac{1}{2}u^2} du, \quad B \in \mathcal{B}(R \setminus \{0\}).$$

Then ρ is a symmetric Lévy measure and

$$\psi(2^{-1}\theta^2) = \int_R (1 - \cos(\theta t))\rho(dt), \quad \theta \in R. \quad (4.1)$$

Let $S = R^T$ and \mathcal{S} be a cylindrical σ -field of R^T with finite measure λ . Define

$$\begin{aligned} F_A(B) &= \lambda(A)\rho(B), \\ F_f(B) &= F\{(s, x) \in R^T \times R : f(s)x \in B \setminus \{0\}\}. \end{aligned}$$

LEMMA 4.2. ([2, Theorem 2.1]) Suppose that $\{X_t, t \in T\}$ is a process of type G without the Gaussian component whose f.d.d.'s are determined by (2.2). Then there is an independently scattered random measure Λ on \mathcal{S} with characteristic function

$$E \exp i\theta\Lambda(A) = \exp\left\{-\int_A \psi(2^{-1}\theta^2)\lambda(ds)\right\}, \quad A \in \mathcal{S}$$

such that

$$\{X_t, t \in T\} \stackrel{d}{=} \left\{\int_{R^T} s(t)\Lambda(ds), t \in T\right\},$$

in the sense of the equality of the f.d.d.'s. Moreover, for any Λ -integrable f ,

$$E \exp i \int_{R^T} f d\Lambda = \exp\left\{-\int_{R^T} \psi(2^{-1}f^2(s))\lambda(ds)\right\}.$$

THEOREM 4.1. (i) For any Λ -integrable $f : R^T \rightarrow R$,

$$E \exp i \int_{R^T} f d\Lambda = \exp\left\{-\int_R (1 - \cos x)\mathcal{F}_f(dx)\right\}.$$

(ii) $f : R^T \rightarrow R$ is Λ -integrable if and only if $\int_R (1 \wedge x^2)\mathcal{F}_f(dx) < \infty$.

Proof. (i) By Lemma 4.2, (4.1) and property of \mathcal{F}_f ,

$$\begin{aligned} E \exp i \int_{R^T} f d\Lambda &= \exp\left\{-\int_{R^T} \psi(2^{-1}f^2(s))\lambda(ds)\right\} \\ &= \exp\left\{-\int_{R^T} \int_R (1 - \cos(xf(s)))\rho(dx)\lambda(ds)\right\} \\ &= \exp\left\{-\int_R (1 - \cos x)\mathcal{F}_f(dx)\right\}. \end{aligned}$$

(ii)

$$\begin{aligned}
f \text{ is } \Lambda\text{-integrable} &\Leftrightarrow \int_{R^T} \psi(2^{-1} f^2(s)) \lambda(ds) < \infty \\
&\Leftrightarrow \int_{R^T} \int_R (1 \wedge x^2 f^2(s)) \rho(dx) \lambda(ds) < \infty \\
&\Leftrightarrow \int_R (1 \wedge x^2) \mathcal{F}_f(dx) < \infty. \quad \square
\end{aligned}$$

Define

$$R(r) = \inf\{x > 0 : \rho([-x, x]^c) \leq r\}.$$

Let $\lambda^{(1)}$ probability measure on R^T such that $\lambda_X \ll \lambda^{(1)}$ and let $h = \frac{d\lambda}{d\lambda^{(1)}}$.

A process $\{X_t, t \in T\}$ is of type G if and only if $\{X_t, t \in T\}$ admits the series representation

$$\{X_t, t \in T\} \stackrel{d}{=} \left\{ \sum_{j=1}^{\infty} \zeta_j^X R_X \left(\frac{\Gamma_j^X}{h_X(V_{j,X})} \right) V_{j,X}(t), t \in T \right\} \quad (4.2)$$

in the sense of equality of f.d.d.'s. In (4.2), the Γ_j 's are the arrival times of a unit rate Poisson process, the ζ_j 's are i.i.d. $N(0, 1)$, the process $\{V_j(t), t \in T\}$ are i.i.d. with the common distribution related to the measure λ . Moreover, the sequences $\{\Gamma_j\}, \{\zeta_j\}, \{V_j(t), t \in T\}$ are independent.

Let $\Delta_{X_{t_{k_1}}, X_{t_{k_2}}}(t_{k_1} < t_{k_2})$ be

$$\sum_{j=1}^{\infty} R^2 \left(\frac{\Gamma_j}{h_{j,X}(V_{j,X})} \right) (V_{j,X}(t_{k_1}) - V_{j,X}(t_{k_2}))^2$$

and consider $\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}$ whose (t_{k_1}, t_{k_2}) -component is $\Delta_{X_{t_{k_1}}, X_{t_{k_2}}}$ and represent

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}} : (R_+^{d(d-1)/2}, \sigma(R_+^{d(d-1)/2})) \rightarrow R$$

as a Lévy measure of $\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}$.

THEOREM 4.2. *Let $\{X_t, t \in T\}$ and $\{Y_t, t \in T\}$ be type G processes with Lévy measure ρ_X, ρ_Y and parameters ψ_X, ψ_Y and λ_X, λ_Y , respectively. Assume $E|X_t| < \infty$ and $E|Y_t| < \infty$. For any d , for any increasing*

Borel set A in $\sigma(R_+^{d(d-1)/2})$ and for any $t_1, t_2, \dots, t_d \in T$,

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}}(A) \geq \rho_{\Delta_{Y_{t_1}, Y_{t_2}, \dots, Y_{t_d}}}(A),$$

then

$$E \sup_{t \in T} X(t) \geq E \sup_{t \in T} Y(t).$$

Proof. Define

$$\begin{aligned} R_X(r) &= \inf\{x > 0 : \rho_X([-x, x]^c) \leq r\}, \\ R_Y(r) &= \inf\{x > 0 : \rho_Y([-x, x]^c) \leq r\}. \end{aligned}$$

Let $\lambda_X^{(1)}$ and $\lambda_Y^{(1)}$ be probability measure on R^T such that $\lambda_X \ll \lambda_X^{(1)}$ and $\lambda_Y \ll \lambda_Y^{(1)}$ and let $h_X = \frac{d\lambda_X}{d\lambda_X^{(1)}}$ and $h_Y = \frac{d\lambda_Y}{d\lambda_Y^{(1)}}$.

Then X_t and Y_t have the following series representation

$$\{X_t, t \in T\} \stackrel{d}{=} \left\{ \sum_{j=1}^{\infty} \zeta_j^X R_X \left(\frac{\Gamma_j^X}{h_X(V_{j,X})} \right) V_{j,X}(t), t \in T \right\}, \quad (4.3)$$

in the sense of equality of f.d.d.'s. Using a similar notation,

$$\{Y_t, t \in T\} \stackrel{d}{=} \left\{ \sum_{j=1}^{\infty} \zeta_j^Y R_Y \left(\frac{\Gamma_j^Y}{h_Y(V_{j,Y})} \right) V_{j,Y}(t), t \in T \right\}. \quad (4.4)$$

Let \mathcal{F}_X and \mathcal{F}_Y be the σ -fields generated on the corresponding sample spaces by $\{\Gamma_j^X\}_{j=1}^{\infty}$ and $\{V_{j,X}(t)\}_{j=1}^{\infty}$ and by $\{\Gamma_j^Y\}_{j=1}^{\infty}$ and $\{V_{j,Y}(t)\}_{j=1}^{\infty}$, respectively. Let \tilde{X}_t and \tilde{Y}_t denote the right-hand sides of (4.3) and (4.4), respectively. Moreover, denoting by $E_{\mathcal{F}_X}$ ($E_{\mathcal{F}_Y}$) the conditional expectation given \mathcal{F}_X (\mathcal{F}_Y), we obtain

$$E_{\mathcal{F}_X}(\tilde{X}(t_{k_1}) - \tilde{X}(t_{k_2}))^2 = \sum_{j=1}^{\infty} R_X^2 \left(\frac{\Gamma_j^X}{h_X(V_{j,X})} \right) (V_{j,X}(t_{k_1}) - V_{j,X}(t_{k_2}))^2,$$

$$E_{\mathcal{F}_Y}(\tilde{Y}(t_{k_1}) - \tilde{Y}(t_{k_2}))^2 = \sum_{j=1}^{\infty} R_Y^2 \left(\frac{\Gamma_j^Y}{h_Y(V_{j,Y})} \right) (V_{j,Y}(t_{k_1}) - V_{j,Y}(t_{k_2}))^2,$$

for any $t_{k_1} < t_{k_2}$. By [7, Theorem 3.1], we know that for any $A \in \sigma(R_+^{d(d-1)/2})$ and for any $t_1, t_2, \dots, t_d \in T$,

$$\rho_{\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}}(A) \geq \rho_{\Delta_{Y_{t_1}, Y_{t_2}, \dots, Y_{t_d}}}(A)$$

which implies $\Delta_{X_{t_1}, X_{t_2}, \dots, X_{t_d}}$ dominates stochastically $\Delta_{Y_{t_1}, Y_{t_2}, \dots, Y_{t_d}}$.

Thus,

$$\begin{aligned} E \sup_{t \in T} X(t) &\geq E \max_{i=1,2,\dots,d} X(t_i) = E \max_{i=1,2,\dots,d} \tilde{X}(t_i) \\ &\geq E \max_{i=1,2,\dots,d} \tilde{Y}(t_i) = E \max_{i=1,2,\dots,d} Y(t_i), \end{aligned}$$

for any $t_1, t_2, \dots, t_d \in T$. □

References

- [1] T.S. Ferguson and M.J. Klass, *A representation of independent increment processes without Gaussian components*, Ann. Math. Statist. 43 (1972), 1634-1643.
- [2] P.S. Kokoszka and M.S. Taqqu, *A characterization of mixing processes of type G*, J. Theoretical Probab. 9 (1996), 3-17.
- [3] B.S. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, Prob. Th. Rel. fields 82 (1989), 451-487.
- [4] J. Robinski, *On path properties of certain infinitely divisible processes*, Stoch. Proc. Appl. 33 (1989), 73-87.
- [5] J. Robinski, *On series representations of infinitely divisible random vectors*, Ann. Prob. 18, 1 (1990), 405-430.
- [6] J. Robinski and G. Samorodnitsky, *Distributions of subadditive functionals of sample paths of infinitely divisible processes*, Ann. Prob. 21, 2 (1993), 996-1014.
- [7] G. Samorodnitsky and M.S. Taqqu, *Stochastic monotonicity and slepian type inequalities for infinitely divisible and stable random vectors*, Ann. Prob. 21, 1 (1993), 143-160.

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