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ALMOST PERIODIC POINTS FOR MAPS OF THE CIRCLE

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Abstract. In this paper, we show that for any continuous map f of the circle S^1 to itself, (1) if $x \in \Omega(f) \setminus \overline{R(f)}$, then x is not a turning point of f and (2) if $P(f)$ is non-empty, then $R(f)$ is closed if and only if $AP(f)$ is closed.

1. Introduction

Let X be a compact metric space, S^1 the unit circle and I the unit closed interval. Suppose that f is a continuous map of X to itself. For any positive integer n, we define $f^1 = f$ and $f^{n+1} = f \circ f^n$. Let f^0 be the identity map of X. Let $AP(f), P(f), R(f), \Gamma(f), \Lambda(f)$ and $\Omega(f)$ denote the set of almost periodic points, periodic points, recurrent points, γ-limit points, ω-limit points and nonwandering points of f, respectively.

In 1980, Z. Nitecki [5] proved that for any piecewise monotone map f of the closed interval I to itself, if $x \in \Omega(f) \setminus \overline{R(f)}$, then $f^n(x)$ is not a turning point of f for any $n \geq 0$. And J.C. Xiong [4] proved that for any continuous map f of the closed interval I itself, $R(f)$ is closed if and only if $AP(f)$ is closed. L. Block, E. Coven, I. Mulvey and Z. Nitecki[7] proved that if f is a continuous map of the circle S^1 to itself such that $P(f)$ is closed and non-empty, then $P(f) = \Omega(f)$. Also, J.S.Bae, S.H.Cho, K.J.Min and S.K. Yang[6] proved that for any continuous map f of the circle if $P(f)$ is empty, then $R(f) = \Omega(f)$.

In this paper, we show that for any continuous map f of the circle S^1 to itself, (1) if $x \in \Omega(f) \setminus \overline{R(f)}$, then x is not a turning point of f and (2) if $P(f)$ is non-empty, then $R(f)$ is closed if and only if $AP(f)$ is closed.

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2. Preliminaries and Definitions

Suppose that f is a continuous map of the circle S^1 to itself. Let $\mathbb{R}$ be the set of real number and $\mathbb{Z}$ be the set of integer. Formally, we think of the circle S^1 as $\mathbb{R} \setminus \mathbb{Z}$. Let $\pi : \mathbb{R} \to \mathbb{R} \setminus \mathbb{Z}$ be the canonical projection. In fact, the map $\pi : \mathbb{R} \to S^1$ is a covering map. We say that a continuous map F from R into itself is a lifting of f if $f \circ \pi = \pi \circ F$. We use the following notations in this paper. Let $a, b \in S^1$ with $a \neq b$, and let $A \in \pi^{-1}(a), B \in \pi^{-1}(b)$ with $|A - B| < 1$ and $A < B$. Then we write $\pi((A, B)), \pi([A, B]), \pi([A, B))$ and $\pi((A, B])$ to denote the open, closed and half-open arcs from a counterclockwise to b, respectively, and we denote it by $(a, b), [a, b], [a, b)$ and $(a, b]$. For $x, y \in [a, b]$ with $a \neq b$. let $X \in \pi^{-1}(x)$, $Y \in \pi^{-1}(y)$ with $X, Y \in [A, B]$, then we define $x > y$ if and only if $X > Y$. In particular, for $a, b, c \in S^1, a < b < c$ means that $b \in (a, c)$. Define a metric d on the circle S^1 by $d(\pi(X), \pi(Y)) = |X - Y|$, where $X, Y \in \mathbb{R}$ and $|X - Y| < \frac{1}{2}$. Then d is a well-defined metric on S^1 which is equivalent to the original one. For the convenience, we use this metric d on S^1.

Let f be a continuous map of the circle S^1 to itself. A point $x \in S^1$ is a periodic point of f provided that for some positive integer $n, f^n(x) = x$. The period of x is the least such integer n. We denote the set of periodic point of f by $P(f)$.

A point $x \in S^1$ is a recurrent point of f provided that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by $R(f)$.

A point $x \in S^1$ is called a nonwandering point of f provided that for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \bigcap U \neq \emptyset$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $y \in S^1$ is called an ω-limit point of $x \in S^1$ provided that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x, f)$. We define $\Lambda(f) = \bigcup_{x \in S^1} \omega(x, f)$ and $\Lambda(A) = \bigcup_{x \in A} \omega(x, f)$ for any subset $A \subset S^1$. Note that $\Lambda(A) \subset \Lambda(B)$ for subsets A, B of S^1 with $A \subset B$.

A point $y \in S^1$ is called an α -limit point of $x \in S^1$ if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points in S^1 with $y_i \to y$ such that $f^{n_i}(y_i) = x$ for all $i \geq 1$. We denote the set of α -limit points of x by $\alpha(x, f)$.

A point $y \in S^1$ is called an γ-limit point of $x \in S^1$ provided that $y \in \omega(x,f) \bigcap \alpha(x,f)$. We denote the set of α-limit points of x by $\gamma(x,f)$ and $\Gamma(f) = \bigcup_{x \in S^1} \gamma(x,f)$.

Now, we define $\alpha_+(x,f)$ and $\alpha_-(x,f)$ as follows : $y \in \alpha_+(x,f)$ (resp., $y \in \alpha_-(x,f)$) provided that there exist a sequence $\{n_i\}$ of positive integer with $n_i \to \infty$ and a sequence $\{y_i\}$ of points in S^1 with $y_i \to y$ such that $f^{n_i}(y_i) = x$ for all $i \geq 1$ and $y < \cdots < y_{i+1} < y_i < \cdots < y_2 < y_1$ (resp., $y_1 < y_2 < \cdots < y_i < y_{i+1} < y$). It is easy to show that if $x \notin P(f)$, then $\alpha(x,f) = \alpha_+(x,f) \bigcup \alpha_-(x,f)$.

A point $x \in S^1$ is called a turning point of f if f is not local homeomorphism at x.

A point x is almostic periodic point of f provided that for any $\epsilon > 0$ one can find an integer $n > 0$ with the following property that for any integer $q > 0$ there exists an integer r with $q \leq r < q + n$ such that $d(f^r(x), x) < \epsilon$, where d is the metric of S^1.

3. Main Results

The following lemmas appear in [1].

Lemma 1. [1] *Suppose that f is a continuous map of the circle S^1 to itself. Then*

$$P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f).$$

Lemma 2. *[1] Let $f \in C^0(S^1, S^1)$ and $J = [a,b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $J \cap P(f) = \emptyset$.*

(a) Suppose that there exists $x \in J$ such that $f(x) \in J$ and $x < f(x)$. Then

(1) if $y \in J, x < y$ and $f(y) \notin [y,b]$, then $[x,y]$ f-covers $[f(x), b]$,

(2) if $y \in J, x > y$ and $f(y) \notin [y,b]$, then $[y,x]$ f-covers $[f(x), b]$.

(b) Suppose that there exists $x \in J$ such that $f(x) \in J$ and $x > f(x)$. Then

(1) if $y \in J, x < y$ and $f(y) \notin [a,y]$, then $[x,y]$ f-covers $[a, f(x)]$,

(2) if $y \in J, y < x$ and $f(y) \notin [a,y]$, then $[y,x]$ f-covers $[a, f(x)]$.

The following lemma appears in [4].

Lemma 3. [4] *Suppose that f is a continuous map of the circle S^1 to itself. Then $x \in AP(f)$ if and only if $x \in \omega(x, y)$ and $\omega(x, f)$ is minimal.*

Proposition 4. *Suppose that f is a continuous map of the circle S^1 to itself. Then*

$$P(f) \subset AP(f) \subset R(f).$$

Proof. By Lemma 3, $AP(f) \subset R(f)$. If $P(f) = \emptyset$, then obviously, $P(f) \subset AP(f)$. Suppose that $P(f) \neq \emptyset$. Let $x \in P(f)$ and n be the period of x. Then $x \in \omega(x, f)$ and $f^n(x) = x$. Let y be any point in $\omega(x, f)$. Then there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. Since $f^n(f^{n_i}(x)) = f^{n+n_i}(x) = f^{n_i+n}(x) = f^{n_i}(f^n(x)) = f^{n_i}(x)$ for all positive integers i, $f^{n_i}(x) \to f^n(y)$. Therefore $y \in P(f)$ and $y \in R(f)$ by Lemma 1. Hence $y \in \omega(y, f)$. Therefore $\omega(x, f) \subset \omega(y, f)$. We show that $\omega(y, f) \subset \omega(x, f)$. Let $z \in \omega(y, f)$. Then there exists a sequence $\{m_i\}$ of positive integer with $m_i \to \infty$ such that $f^{m_i} \to z$. Since $y \in \omega(x, f)$ and $f^{n_i}(x) \to y$, $f^{m_i+n_i}(x) \to z$. Hence $z \in \omega(x, f)$. Thus $\omega(y, f) \subset \omega(x, f)$. Therefore $\omega(x, f)$ is a minimal set. Hence we have $x \in AP(f)$ by Lemma 3. The proof is completed. □

By combining Lemma 1 and Proposition 4, we have the following proposition.

Proposition 5. *Suppose that f is a continuous map of the circle S^1 to itself. Then $P(f) \subset AP(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f)$.*

Lemma 6. *[1] Suppose that f is a continuous map of the circle S^1 to itself. Then $x \in \Omega(f)$ if and only if $x \in \alpha(x, f)$.*

Theorem 7. *Let f be a continuous map of the circle S^1 to itself. If $x \in \Omega(f) \setminus \overline{R(f)}$, then x is a not turning point of f.*

Proof. Suppose x is a turning point of f. Let C be a connected component of $S^1 \setminus \overline{R(f)}$ containing x. Then there exist $a, b \in C$ with $a \neq b$ such that $x \in (a, b)$, $(a, b) \bigcap P(f) = \emptyset$ and $f^n \notin (a, b)$ for all $n \geq 1$. Since $x \in \Omega(f), x \in \alpha(x, f)$ by Lemma 6. Without loss of generality, we may assume that $x \in \alpha_+(x, f)$. Then there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{x_i\}$

of points in S^1 with $x_i \to x$ such that $f^{n_i}(x_i) = x$ for all $i \geq 1$ and $a < x < \cdots < x_i < b$. Since x is a turning point of f, there exists a point $z \in (a, x)$ such that $f(z) = f(x_i)$ for sufficiently large i. Hence $x = f^{n_i}(x_i) = f^{n_i}(z) > z$. By Lemma 4,

$$[x, x_i] \quad f^{n_i}\text{- covers} \quad [a, x]$$

and

$$[z, x] \quad f^{n_i}\text{- covers} \quad [x, b].$$

In particular, $[x, x_i]$ f^{n_i}- covers $[z, x]$ and $[z, x]$ f^{n_i}- covers $[x, x_i]$. Therefore $[x, x_i]$ f^{n_i}-covers itself. Hence f has a periodic point in (a, b),a contradiction. The proof is completed. □

Proposition 8. *Suppose that f is a continuous map of the circle S^1 to itself. Then $\Lambda(\overline{R(f)}) \subset \Lambda(\Omega(f)) \subset \Gamma(f)$.*

Proposition 9. *Let f be a continuous map of the circle S^1 to itself. If $R(f)$ is closed, then $R(f) = AP(f)$. Thus $AP(f) = R(f) = \Gamma(f) = \overline{R(f)}$.*

Proof. We know that $AP(f) \subset R(f)$ by Proposition 5. Hence we show that $R(f) \subset AP(f)$. Let $x \in R(f)$. Then $x \in \omega(x, f)$. We show that $\omega(x, f)$ is minimal. Let y be arbitrary point in $\omega(x, f)$. Then there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. Suppose that z is any point in $\omega(x, f)$. Then there exists a sequence $\{m_i\}$ of positive integers with $m_i \to \infty$ such that $f^{m_i}(y) \to z$. Therefore $f^{m_i+n_i}(x) \to z$. Hence $z \in \omega(x, f)$. Thus $\omega(x, f) \supset \omega(y, f)$. Since y is arbitrary point in $\omega(x, f)$, it suffices to show that $y \in \omega(y, f)$. Since $x \in R(f)$, $y \in \omega(x, f) \subset \Lambda(R(f)) \subset \overline{R(f)})$. By Proposition 8, $y \in \Gamma(f)$. Since $R(f)$ is closed, $y \in R(f)$. Therefore $y \in \omega(y, f)$. Hence $\omega(x, f) \subset \omega(y, f)$. Therefore $\omega(x, f)$ is minimal. By Lemma 3, $x \in AP(f)$. Therefore $R(f) \subset AP(f)$. The proof is completed. □

Lemma 10. *[2] Suppose that f is a continuous map of the circle S^1 to itself, and $P(f) \neq \emptyset$. Then $\overline{P(f)} = \overline{R(f)}$.*

Theorem 11. *Suppose that f is a continuous map of the circle S^1 to itself and $P(f) \neq \emptyset$. Then $R(f)$ is closed if and only if $AP(f)$ is closed.*

Proof. Suppose that $AP(f)$ is closed. Then we know $AP(f) = \overline{P(f)}$. By Lemma 10, we have $AP(f) = \overline{R(f)}$. Also by Proposition 5, $AP(f) = R(f) = \overline{R(f)}$. Therefore $R(f)$ is closed. Assume that $R(f)$ is closed. Then $R(f) = AP(f)$ by Proposition 9. Therefore $AP(f)$ is closed. The proof is completed. □

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