

SOME CONDITIONS FOR COMAXIMALITY OF IDEALS

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ABSTRACT. In this paper, it is shown that if R is a commutative ring with identity and there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$.

1. Introduction

Throughout this paper, R will be a commutative ring with identity and I, J and I_1, I_2, \dots, I_n are ideals of R , unless otherwise stated. If S is a multiplicatively closed subset of R , then R_S is the quotient ring of R with respect to S and $I R_S$ is the extension of I in R_S . $Ass_R(R/I)$ denotes the set of all associated prime ideals of I . We denote by $(I :_R J) = \{ r \in R : rJ \subseteq I \}$ the annihilator of $(J+I)/I$. It is well known that if I_1, I_2, \dots, I_n are pairwise comaximal ideals of R , then $I_1 \cap I_2 \cap \cdots \cap I_n = I_1 I_2 \cdots I_n$. In [3], Ratliff gave some variations of this. He showed that if I_1, I_2, \dots, I_n are ideals of a Noetherian ring R , then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if each $P \in Ass_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n . The purpose of this paper is to generalize this result for ideals in a commutative ring with identity.

In section 2, it is shown that if R is a commutative ring with identity and there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R$

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$I_1 \cdots I_{i-1} I_{i+1} \cdots I_n$). It is also shown that if $I_1 I_2 \cdots I_n$ has a finite primary decomposition and each I_i is finitely generated, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal and

$$S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset.$$

Finally, we show that if I_1, I_2, \dots, I_n are ideals in a Noetherian ring R , then each $P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n if and only if $(R/(I_1 I_2 \cdots I_n))_S \cong \prod_{j=1}^n (R/I_j)_{S_j}$, where

$$S = R \setminus \bigcup \{P : P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n)\}$$

and

$$S_j = R \setminus \bigcup \{P : P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n) \text{ and } I_j \in P\}.$$

2. Main Results

We begin this section by listing some of known results that will be used in this paper.

LEMMA 2.1. *Let R be a commutative ring with identity, $I_1, I_2, \dots, I_n, I, J$ and K ideals of R and S a multiplicatively closed subset of R . Then*

- (1) $(IJ)R_S = IR_S J R_S$.
- (2) $(I \cap J)R_S = IR_S \cap J R_S$.
- (3) $(I :_R J)R_S \subseteq (IR_S :_{R_S} J R_S)$.

If J is finitely generated then the equality holds.

- (4) $I \subseteq (IR_S) \cap R$.

The equality holds if and only if $S \cap Z(R/I) = \emptyset$, where $Z(R/I) = \{x \in R : (I :_R x) \neq I\}$.

- (5) *If I has a finite primary decomposition then $Z(R/I) = \cup \{P : P \in \text{Ass}_R(R/I)\}$.*

- (6) $(I_1 \cap I_2 \cap \cdots \cap I_n :_R J) = \bigcap_{i=1}^n (I_i :_R J)$.
- (7) $(J :_R I_1 + \cdots + I_n) = \bigcap_{i=1}^n (J :_R I_i)$.
- (8) If Q is a primary component of I then $I \subset (I :_R Q)$.
- (9) $(I :_R JK) = ((I :_R J) :_R K)$.

Proof. This is straightforward. \square

The following results concerning pairwise comaximal ideals are well known.

LEMMA 2.2. *Let R be a commutative ring with identity and let I_1, I_2, \dots, I_n be pairwise comaximal ideals of R . Then*

- (1) *For each i , I_i and $(I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m$ are comaximal, for all positive integer m .*
- (2) *For each i , $(I_i :_R (I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m) = I_i$, for all positive integer m .*
- (3) $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R (I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)^m)$,
for all positive integer m .

Proof. It is easy to see that if I and J are comaximal then $(I :_R J) = I$. Hence (2) follows from (1). To prove (1), it suffices to show that if I, J and K are pairwise comaximal ideals then I and JK are comaximal. Since $R = (I+J)(I+K) = I^2 + IJ + IK + JK \subseteq I + JK$, hence I and JK are comaximal. For (3), the last equality follows from (1). To show the first equality, it may be assumed that $n = 2$. Let $x \in I_1 \cap I_2$. Since I_1 and I_2 are comaximal, there exist $a \in I_1, b \in I_2$ such that $a + b = 1$. $x = ax + bx \in I_1 I_2$. Thus $I_1 \cap I_2 \subseteq I_1 I_2$. The opposite inclusion is clear. Hence $I_1 I_2 = I_1 \cap I_2$. \square

The following is the Chinese remainder theorem which is well known, so we omit the proof.

THEOREM 2.3. *Let R be a commutative ring with identity and let I_1, I_2, \dots, I_n be pairwise comaximal ideals of R . If b_1, b_2, \dots, b_n are elements of R , then there exists $b \in R$ such that $b \equiv b_i \pmod{I_i}$, for $i = 1, 2, \dots, n$.*

LEMMA 2.4. *Let R be a commutative ring with identity and let I_1, I_2, \dots, I_n be pairwise comaximal ideals of R . Then A map $f : R \longrightarrow \prod_{j=1}^n R/I_j$ defined by $f(r) = (r + I_1, r + I_2, \dots, r + I_n)$ is an epimorphism of rings if and only if I_1, I_2, \dots, I_n are pairwise comaximal. In this case, $R/(I_1 I_2 \cdots I_n) \cong \prod_{j=1}^n R/I_j$.*

Proof. Let the map $f : R \longrightarrow \prod_{j=1}^n R/I_j$ defined by $f(r) = (r + I_1, r + I_2, \dots, r + I_n)$ be an epimorphism of rings. For any $i \in \{1, 2, \dots, n\}$, by resubscripting, if necessary, we may assume that $i = 1$. Since f is onto, there exists $r \in R$ such that $f(r) = (1 + I_1, I_2, \dots, I_n)$. Thus $r - 1 \in I_1$ and $r \in I_2 \cap I_3 \cap \cdots \cap I_n$. This implies that there exists $a \in I_1$ such that $r - 1 = a$. Thus $1 = r - a \in I_1 + I_j$, for $j = 2, 3, \dots, n$. Hence I_1, I_2, \dots, I_n are pairwise comaximal. The converse follows from the Chinese remainder theorem. \square

Theorem(2.5) is one of our main theorems in this paper.

THEOREM 2.5. *Let R be a commutative ring with identity and let I_1, I_2, \dots, I_n be ideals of R . If there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal, then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$.*

Proof. Let S be a multiplicatively closed subset of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal. By lemma(2.1)(4), $(I_1 I_2 \cdots I_n) R_S \cap R = I_1 I_2 \cdots I_n$. Since $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal,

$$\begin{aligned} I_1 R_S I_2 R_S \cdots I_n R_S &= I_1 R_S \cap I_2 R_S \cap \cdots \cap I_n R_S \\ &= \bigcap_{i=1}^n (I_i R_S :_{R_S} I_1 R_S \cdots I_{i-1} R_S I_{i+1} R_S \cdots I_n R_S). \end{aligned}$$

By contracting these ideals to R , we get

$$I_1 I_2 \cdots I_n \subseteq I_1 \cap I_2 \cap \cdots \cap I_n \subseteq (I_1 R_S \cap I_2 R_S \cap \cdots \cap I_n R_S) \cap R$$

$$= (I_1 R_S I_2 R_S \cdots I_n R_S) \cap R = (I_1 I_2 \cdots I_n) R_S \cap R = I_1 I_2 \cdots I_n$$

and

$$\begin{aligned} I_1 I_2 \cdots I_n &\subseteq I_1 \cap I_2 \cap \cdots \cap I_n \\ &\subseteq \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n) \\ &\subseteq \left(\left(\bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n) \right) R_S \right) \cap R \\ &= \left(\bigcap_{i=1}^n \left((I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n) R_S \right) \right) \cap R \\ &\subseteq \left(\bigcap_{i=1}^n (I_i R_S :_{R_S} I_1 R_S \cdots I_{i-1} R_S I_{i+1} R_S \cdots I_n R_S) \right) \cap R \\ &= (I_1 R_S \cap I_2 R_S \cap \cdots \cap I_n R_S) \cap R = I_1 I_2 \cdots I_n. \end{aligned}$$

Thus $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$. \square

As an application of the result in [3], Ratliff proved the following theorem. This gives an example of ideals for which the conditions in theorem(2.5) are satisfied. We refer the reader to [3, theorem(2.7)] for a proof.

THEOREM 2.6. *Let R be a Noetherian ring and a_1, a_2, \dots and a_n nonunit elements on R . Let each permutation of a_1, a_2, \dots and a_n be an R -sequence. If I_1, I_2, \dots and I_m are ideals generated by disjoint subset of a_1, a_2, \dots and a_n , then $I_1^{k_1} I_2^{k_2} \cdots I_m^{k_m} = I_1^{k_1} \cap I_2^{k_2} \cap \cdots \cap I_m^{k_m} = \bigcap_{j=1}^m (I_j^{k_j} :_R (I_1 \cdots I_{j-1} I_{j+1} \cdots I_m)^t)$, for all positive integer k_1, \dots, k_m and t .*

Theorem(2.7) is another main theorem in this paper.

THEOREM 2.7. *Let R be a commutative ring with identity and let I_1, I_2, \dots, I_n be finitely generated ideals in R . Let $I_1 I_2 \cdots I_n$ have a finite primary decomposition. Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal and $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$.*

Proof. Assume that

$$\begin{aligned} I_1 I_2 \cdots I_n &= I_1 \cap I_2 \cap \cdots \cap I_n \\ &= \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n). \end{aligned}$$

Let $S = R \setminus Z(R/(I_1 I_2 \cdots I_n))$. Then S is a multiplicatively closed subset of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$. Since $I_1 I_2 \cdots I_n$ has a finite primary decomposition, it follows from lemma(2.4)(5) that $Z(R/(I_1 I_2 \cdots I_n)) = \cup \{ P : \text{Ass}_R(R/(I_1 I_2 \cdots I_n)) \}$. Thus, to show that $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal, it suffices to show that no $P \in \text{Ass}_R(R/(I_1 I_2 \cdots I_n))$ contains more than one of the I_j . For any $i \neq k \in \{1, 2, \dots, n\}$, by resubscripting, if necessary, we may assume that $i = 1, k = 2$. Then we have

$$\begin{aligned} I_1 \cap I_2 \cap \cdots \cap I_n &\subseteq (I_1 \cap I_2 \cap \cdots \cap I_n :_R I_1 + I_2) = \bigcap_{i=1}^n (I_i :_R I_1 + I_2) \\ &= (I_1 :_R I_2) \cap (I_2 :_R I_1) \cap (I_3 :_R I_1 + I_2) \cap \cdots \cap (I_n :_R I_1 + I_2) \\ &\subseteq (I_1 :_R I_2) \cap (I_2 :_R I_1) \cap (I_3 :_R I_1 I_2) \cap \cdots \cap (I_n :_R I_1 I_2) \\ &\subseteq \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n) \\ &= I_1 \cap I_2 \cap \cdots \cap I_n. \end{aligned}$$

This show that, for all $i \neq k \in \{1, 2, \dots, n\}$,

$$I_1 \cap I_2 \cap \cdots \cap I_n = (I_1 \cap I_2 \cap \cdots \cap I_n :_R I_i + I_k).$$

By hypothesis, for all $i \neq k \in \{1, 2, \dots, n\}$,

$$I_1 I_2 \cdots I_n = (I_1 I_2 \cdots I_n :_R I_i + I_k).$$

Hence $I_1 I_2 \cdots I_n = (I_1 I_2 \cdots I_n :_R (I_i + I_k)^m)$, for all $m \geq 1$ and for all $i \neq k \in \{1, 2, \dots, n\}$. If $P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n)$ such that $I_i + I_k \subseteq P$, then $(I_i + I_k)^m$ is contained in a P -primary component Q of $I_1 I_2 \cdots I_n$, for large m . This implies that

$$\begin{aligned} I_1 I_2 \cdots I_n &\subset (I_1 I_2 \cdots I_n :_R Q) \\ &\subseteq (I_1 I_2 \cdots I_n :_R (I_i + I_k)^m) = I_1 I_2 \cdots I_n, \end{aligned}$$

which is a contradiction. Hence $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal. The converse follows from theorem(2.7). \square

If R is a Noetherian ring then each ideal has a finite primary decomposition. Hence we have the following corollary.

COROLLARY 2.8. *Let R be a Noetherian ring and let I_1, I_2, \dots, I_n be ideals in R . Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if there exists a multiplicatively closed subset S of R such that $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal and $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$.*

Proof. This immediately follows from theorem(2.7). \square

The following theorem is proved by Ratliff in [3].

THEOREM 2.9. *Let R be a Noetherian ring and let I_1, I_2, \dots, I_n be ideals in R . Then $I_1 I_2 \cdots I_n = I_1 \cap I_2 \cap \cdots \cap I_n = \bigcap_{i=1}^n (I_i :_R I_1 \cdots I_{i-1} I_{i+1} \cdots I_n)$ if and only if each $P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n .*

COROLLARY 2.10. *Let R be a Noetherian ring and let I_1, I_2, \dots, I_n be ideals in R . Then there exists a multiplicatively closed subset S of R such that $S \cap Z(R/(I_1 I_2 \cdots I_n)) = \emptyset$ and $I_1 R_S, I_2 R_S, \dots, I_n R_S$ are pairwise comaximal if and only if each $P \in \text{Ass}_R(R/I_1 I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n .*

Proof. This follows from corollary(2.8) and theorem(2.9). \square

The following corollary is analogous to lemma(2.4).

COROLLARY 2.11. *Let R be a Noetherian ring and let I_1, I_2, \dots, I_n be ideals in R . Then each $P \in \text{Ass}_R(R/I_1I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n if and only if $(R/(I_1I_2 \cdots I_n))_S \cong \prod_{j=1}^n (R/I_j)_{S_j}$, where*

$$S = R \setminus \bigcup \{P : P \in \text{Ass}_R(R/I_1I_2 \cdots I_n)\}$$

and

$$S_j = R \setminus \bigcup \{P : P \in \text{Ass}_R(R/I_1I_2 \cdots I_n) \text{ and } I_j \in P\}.$$

Proof. By corollary(2.10), each $P \in \text{Ass}_R(R/I_1I_2 \cdots I_n)$ contains exactly one of the ideals I_1, I_2, \dots, I_n if and only if $I_1R_S, I_2R_S, \dots, I_nR_S$ are pairwise comaximal. Thus, by lemma(2.4),

$$R_S/((I_1I_2 \cdots I_n)R_S) \cong \prod_{j=1}^n R_S/(I_jR_S)$$

It is easy to see that $R_S/(I_jR_S) \cong R_{S_j}/(I_jR_{S_j})$ and $(R/I)_T \cong R_T/(IR_T)$ for all ideals I and for all multiplicatively closed subset T of R . This completes the proof. \square

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