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ON THE C_1-CONSTRUCTION

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Abstract. In [4], it is proved that the algebra (S, k^{14}) is not a C_1-construction if the field is the real number field. In this paper, we will introduce a different proof of the fact that the algebra (S, k^{14}) is not a C_1-construction.

1. Introduction

Throughout this paper, k will denote the real number field and $M_n(k)$ will denote the set of all $n \times n$ matrices with entries in k. A commutative k-subalgebra R of $M_n(k)$ is a maximal, commutative k-subalgebra of $M_n(k)$ if and only if R satisfies the following condition: If R^* is a commutative k-subalgebra of $M_n(k)$ with $R \subset R^*$, then $R = R^*$. Let $\mathbf{X}$ denote the category whose objects are ordered pairs (G, H), where G is a finite dimensional, local, commutative k-algebra and H is a finitely generated, faithful G-module. Let $(B, M) \in \mathbf{X}$. The direct sum $B \oplus M$ of the B-modules B and M can be given the structure of a commutative k-algebra by defining multiplication in the following way.

$$(b_1, m_1)(b_2, m_2) = (b_1 b_2, m_2 b_1 + m_1 b_2), b_i \in B, m_i \in M, i = 1, 2.$$

The commutative ring thus defined is called the idealization of M and will be denoted by $B \ltimes M$.

Definition 1.1. Suppose R is a maximal commutative k-subalgebra of $M_n(k)$. We say R is a (B, N)-construction if R is k-algebra isomorphic to $B \ltimes N^\ell$ for some $(B, N) \in \mathbf{X}$ and a positive integer ℓ.

Here, N^ℓ denotes the direct sum of ℓ copies of B-module N.

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Remark 1.2. The Courter's algebra R in [1] is a (B, N)-construction.

The B-module $B^\ell \oplus N$ is a $B \ltimes N^\ell$-module with scalar multiplication defined as follows.

$$(b_1, \ldots, b_\ell, n)(b, n_1, \ldots, n_\ell) = (b_1 b, \ldots, b_\ell b, nb + \sum_{i=1}^{\ell} n_i b_i).$$

Remark 1.3. For $(B, N) \in \mathbf{X}$, it is known that $B^\ell \oplus N$ is a finitely generated, faithful, $B \ltimes N^\ell$-module.

If $(G, H), (G', H')$ are two objects in $\mathbf{X}$, then a morphism from (G, H) to (G', H') is an ordered pair (σ, τ), where $\sigma : G \longrightarrow G'$ is a k-algebra homomorphism, $\tau : H \longrightarrow H'$ is a k-vector space homomorphism with $\tau(hg) = \tau(h)\sigma(g)$ for all $h \in H$ and $g \in G$. We will use the notation $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ to indicate the morphism (σ, τ) from (G, H) to (G', H'). We call a morphism $(\sigma, \tau) : (G, H) \longrightarrow (G', H')$ an isomorphism if σ is a k-algebra isomorphism and τ is a k-vector space isomorphism. In this case we will use the notation $(G, H) \cong_{(\sigma,\tau)} (G', H')$.

Definition 1.4. With the above notations, $(G, H) \in \mathbf{X}$ is a C_1-construction if $(G, H) \cong_{(\sigma,\tau)} (B \ltimes N^\ell, B^\ell \oplus N)$ for some $(B, N) \in \mathbf{X}$ and a positive integer ℓ.

Remark 1.5. If R is the Courter's algebra in [1], then (R, k^{14}) is a C_1-construction.

2. Main results

As we have proved in [4], the algebra (S, k^{14}) is a maximal commutative subalgebra of matrix algebra of size 14 which is not isomorphic to the Courter's algebra. Recall that the element r in the Jacobson radical of S is the following form :

$$\begin{pmatrix}
\multicolumn{2}{l}{O_{2\times 2}} & \multicolumn{10}{c}{O_{2\times 10}} & O_{2\times 2} \\
c_1 & O & & & & & & & & & & & \\
O & c_1 & & & & & & & & & & & \\
c_2 & O & & & & & & & & & & & \\
O & c_2 & & & & & & & & & & & \\
c_3 & O & \multicolumn{10}{c}{O_{10\times 10}} & O_{10\times 2} \\
O & c_3 & & & & & & & & & & & \\
c_4 & O & & & & & & & & & & & \\
O & c_4 & & & & & & & & & & & \\
d_1 & d_2 & & & & & & & & & & & \\
d_3 & d_4 & & & & & & & & & & & \\
e_1 & e_2 & c_1+d_1 & d_2 & c_2+d_3 & d_4 & c_3 & O & c_4 & O & c_1 & c_2 & \\
e_3 & e_4 & O & c_1 & O & c_2 & d_1 & c_3+d_2 & d_3 & c_4+d_4 & c_3 & c_4 & O_{2\times 2}
\end{pmatrix}$$

Here, $c_i, d_i, e_i \in k$ for all $i = 1, 2, 3, 4$ and the element s in S is the followong form :

$$s = r + aI_n$$

for some $a \in k$.

In [4], it is proved the algebra (S, k^{14}) is not a C_1-construction if the field is the real number field. Here, in this section, we will introduce a different proof of the fact that the algebra (S, k^{14}) is not a C_1-construction.

The next theorem can be found in [2] and we restate it.

THEOREM 2.1. *Suppose $(R, J(R), k)$ is a local maximal commutative subalgebra of matrix algebra of size 14, $dim_k R = 13$ and $i(J(R)) = 3$. Then, (R, k^{14}) is a C_1-construction if and only if there exist R-module generators θ_1 and θ_2 of k^{14} whose annihilators $I_1 = Ann_R(\theta_1)$ and $I_2 = Ann_R(\theta_2)$ satisfy the following three properties:*

(1) $(0) :_R I_i = I_1 + I_2$, $i = 1, 2$.

(2) $(0) \to (I_1 + I_2) \to R \to R/(I_1 + I_2) \to (0)$ splits as k-algebras.

(3) There exists an R-module isomorphism $f : I_1 \to I_2$ such that $\theta_1 f(x) = \theta_2 x$ for all $x \in I_1$.

In [4], we have proved the following Lemma.

LEMMA 2.2. *Suppose S is the maximal commutative subalgebra in [4]. Then,*

(1) $dim_k S = 13$, $i(J(S)) = 3$

(2) The minimal number of generators of k^{14} is 2.

(3) The socle of S, $Soc(S)$ is generated by $E_{13,1}, E_{13,2}, E_{14,1}, E_{14,2}$. Here, $E_{i,j}$ is a matrix of size 14 whose entries are all zero except the (i,j)-th entry that is 1.

We now introduce a different proof of the fact that (S, k^{14}) is not a C_1-construction under the real number field.

THEOREM 2.3. *Let k be the real number field. Suppose S is the maximal commutative subalgebra in [4]. Then, (S, k^{14}) is not a C_1-construction.*

Proof. Let $\{\theta_1, \theta_2\}$ be an arbitrary S-module generator. Then, there exist $a_i, b_i \in k, i = 1, \ldots, 14$ such that

$$\theta_1 = \sum_{i=1}^{14} a_i \epsilon_i, \ \theta_2 = \sum_{i=1}^{14} b_i \epsilon_i$$

Here, $\{\epsilon_1, \ldots, \epsilon_{14}\}$ is the standard basis of k^{14}. Now, let $I_i = Ann_S(\theta_i)$ for $i = 1, 2$. If $r \in I_1$, then we can write $r = aI_{14} + r^*$ for some $a \in k$ and $r^* \in J(S)$. Thus, we have

$$\theta_1 a + \theta_1 r^* = 0$$

This implies

$$a_{13} a = 0, a_{14} a = 0$$

Note that $\epsilon_i \in k^{14}J(S)$ for $i = 1, \ldots, 12$ and $dim_k(k^{14}/k^{14}J(S)) = 2$. Thus, $\{a_{13}\epsilon_{13} + a_{14}\epsilon_{14}, b_{13}\epsilon_{13} + b_{14}\epsilon_{14}\}$ is a k-vector space basis of $k^{14}/k^{14}J(S)$. This implies $a_{13} \neq 0$ or $a_{14} \neq 0$ and hence $a = 0$ and so $r = r^*$. Thus, $I_1 \subset J(S)$ and by the similar way, we get $I_2 \subset J(S)$.

Now, we want to show that $I_i \subset Soc(S)$ for $i = 1, 2$ by considering the following three cases :

(case 1) $a_{13} \neq 0, a_{14} \neq 0$
(case 2) $a_{13} \neq 0, a_{14} = 0$
(case 3) $a_{13} = 0, a_{14} \neq 0$

We now consider the first case.
If $r \in I_1$, then $r \in J(S)$ and hence r can be written as follows:

$$\begin{pmatrix}
\multicolumn{2}{c}{O_{2\times 2}} & \multicolumn{10}{c}{O_{2\times 10}} & \multicolumn{2}{c}{O_{2\times 2}} \\
c_1 & O & & & & & & & & & & & & \\
O & c_1 & & & & & & & & & & & & \\
c_2 & O & & & & & & & & & & & & \\
O & c_2 & & & & & & & & & & & & \\
c_3 & O & \multicolumn{10}{c}{\multirow{2}{*}{$O_{10\times 10}$}} & \multicolumn{2}{c}{\multirow{2}{*}{$O_{10\times 2}$}} \\
O & c_3 & & & & & & & & & & & & \\
c_4 & O & & & & & & & & & & & & \\
O & c_4 & & & & & & & & & & & & \\
d_1 & d_2 & & & & & & & & & & & & \\
d_3 & d_4 & & & & & & & & & & & & \\
e_1 & e_2 & c_1+d_1 & d_2 & c_2+d_3 & d_4 & c_3 & O & c_4 & O & c_1 & c_2 & \multicolumn{2}{c}{\multirow{2}{*}{$O_{2\times 2}$}} \\
e_3 & e_4 & O & c_1 & O & c_2 & d_1 & c_3+d_2 & d_3 & c_4+d_4 & c_3 & c_4 & &
\end{pmatrix}$$

Since $\theta_1 r = 0$, we have the following equations :

$$\begin{array}{lll}
a_{13}(c_1+d_1)=0, & a_{13}d_2+a_{14}c_1=0, & a_{13}(c_2+d_3)=0, \\
a_{13}d_4+a_{14}c_2=0, & a_{13}c_3+a_{14}d_1=0, & a_{14}(c_3+d_2)=0, \quad (1) \\
a_{13}c_4+a_{14}d_3=0, & a_{14}(c_4+d_4)=0, & a_{13}c_1+a_{14}c_3=0, \\
a_{13}c_2+a_{14}c_4=0 & &
\end{array}$$

From the above equations we have

$$a_{13}^2c_1-a_{14}^2d_1=0,\ a_{13}^2c_2-a_{14}^2d_3=0,\ a_{13}^2(c_1+d_1)=0,\ a_{13}^2(c_2+d_3)=0$$

Thus, we have

$$(a_{13}^2+a_{14}^2)d_1=0,\ (a_{13}^2+a_{14}^2)d_3=0$$

and hence $d_1=0$ and $d_3=0$. Applying the condition $a_{13}\neq 0$ to the equation (1), we have $c_i=0, d_i=0$ for $i=1,2,3,4$. Thus, r is a linear combination of the matrices $E_{13,1}, E_{13,2}, E_{14,1}, E_{14,2}$ and by Lemma 2.2, $r\in Soc(S)$.

We also can show that I_1 is the suset of $Soc(S)$ in the second case by applying exactly the same methods as the first case.

We now assume $a_{13}=0, a_{14}\neq 0$. Since $a_{13}=0$, from the equation (1) we get the following equations:

$$a_{14}d_1=0,\ a_{14}c_2=0,\ a_{14}d_3=0,\ a_{14}(c_3+d_2)=0,$$

$$a_{14}c_3=0,\ a_{14}(c_4+d_4)=0,\ a_{14}c_1=0,\ a_{14}c_4=0$$

Since $a_{14}\neq 0$, we have $c_i=0, d_i=0$ for $i=1,2,3,4$ and so $r\in Soc(S)$.

Thus, in all of the three cases, we have $I_1 \subset Soc(S)$. Similarly, we can show $I_2 \subset Soc(S)$. This implies $I_1 + I_2 \subset Soc(S)$. Note that $(0) :_S I_i = J(S)$ for $i = 1, 2$ and moreover, by Lemma 2.2, $Soc(S)$ is a proper subset of $J(S)$. Therefore, the algebra S doesn't satisfy the condition (1) in Theorem 2.1 and we now conclude the algebra (S, k^{14}) is not a C_1-construction. □

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