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ON THE RELATION BETWEEN THE TIMEWIDTHS Δ_f AND Δ_{f*h}

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Abstract. In the present paper we shall first introduce the timewidth of a signal, and then we shall investigate the relation between the timewidths of a signal f and of the convolution $f * h$ for some other signal h.

1. Introduction

In this paper we shall consider the signals in the space

$$\mathcal{G} \stackrel{\text{def}}{=} \{f | f \in L^2(\mathbb{R}), tf(t) \in L^2(\mathbb{R})\}.$$

A signal f is said to be band-limited if there is an $\omega_c^f \in \mathbb{R}$ such that

$$\text{supp}(\hat{f}) \subset [-\omega_c^f, \omega_c^f],$$

where we call ω_c^f the off-frequency of f.

The purpose of the present paper is to investigate the relationship between the timewidths of $f \in \mathcal{G}$ and of the convolution $f * h$ for some h.

2. Preliminaries

We first introduce the concept of the timewidth of a signal in the space $\mathcal{G}$;

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Definition 1. *For each $f \in \mathcal{G}$, we define the timewidth Δ_f of f by*

$$\Delta_f = \left(\frac{\int_{\mathbb{R}} (t-\chi_f)^2 |f(t)|^2 dt}{\int_{\mathbb{R}} |f(t)|^2 dt} \right)^{\frac{1}{2}},$$

where

$$\chi_f = \frac{\int_{\mathbb{R}} t|f(t)|^2 dt}{\int_{\mathbb{R}} |f(t)|^2 dt}.$$

Lemma 1. *Let T_a and S_a be operators on $\mathcal{G}$ defined by $T_a f(x) = f(x-a)$ and $S_a f(x) = f(ax)$. Then we have*

(1) $\Delta_{T_a f} = \Delta_f, \quad \Delta_{S_a f} = \frac{1}{a}\Delta_f$.
(2) $S_a(f * h) = |a| S_a f * S_a h$.

Proof. It is easily verified from Definition 1. □

In Lemma 1, (1) states that the timewidth Δ_f of signal f is not only translation invariant but also it is compressed by the ratio of the compression of the variable. So we can assume $\chi_f = 0$, whence we have

$$\Delta_f = \left(\frac{\int_{\mathbb{R}} t^2 |f(t)|^2 dt}{\int_{\mathbb{R}} |f(t)|^2 dt} \right)^{\frac{1}{2}} \tag{1}$$

Throughout the present paper we shall fix an even function h with finite support and with positive essential infinimum. Then we have

$$\chi_{f*h} = 0 \tag{2}$$

for any even signal $f \in \mathcal{G}$. Hence we have, for $F = f * h$,

$$\Delta_F = \int_{\mathbb{R}} t^2 |F(t)|^2 dt$$

from (1) and (2).

In (2) of Lemma 1, put $a = -1$, then we have

$$S_{-1}(f * h) = S_{-1} f * S_{-1} h$$

and hence $S_{-1} f = f$ for even f, $S_{-1} f = -f$ for odd f. Therefore the convolution of even and even or odd and odd is even, but the convolution of even and odd is odd.

We can also assume that the signal $f(t)$ is even, because a lot of signals in a practical problem aren't defined in the interval $(-\infty, 0]$, but they can be extended over $\mathbb{R}$ to be even.

3. Main Theorem

LEMMA 2. *If $f, h \in \mathcal{G}$, then $f * h \in \mathcal{G}$ and*

$$[t(f*h)(t)]^\wedge(\omega) = \sqrt{2\pi}\left([\tau f(\tau)]^\wedge(\omega)\cdot\hat{h}(\omega) + [\tau h(\tau)]^\wedge(\omega)\cdot\hat{f}(\omega)\right).$$

Proof.

$$\begin{aligned}
&[t(f*h)(t)]^\wedge(\omega)\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} t\left(\int_{\mathbb{R}} f(\tau)h(t-\tau)d\tau\right)e^{-i\omega t}dt\\
&= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tau f(\tau)e^{-i\omega\tau}d\tau\int_{\mathbb{R}} h(t)e^{-i\omega t}dt\\
&\quad + \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(\tau)e^{-i\omega\tau}d\tau\int_{\mathbb{R}} th(t)e^{-i\omega t}dt\\
&= \sqrt{2\pi}\left([\tau f(\tau)]^\wedge(\omega)\cdot\hat{h}(\omega) + (\tau h(\tau))^\wedge(\omega)\cdot\hat{f}(\omega)\right). \qquad \square
\end{aligned}$$

THEOREM. *Let $h \in \mathcal{G}$ be an even function having finite support and positive essential infinimum. Then, for any $f \in \mathcal{G}$, we have*

$$\Delta_{f*h} \le \sqrt{2\pi}\left(\frac{M_h}{m_h} + \frac{M_f\|h\|\Delta_h}{m_h\|f\|\Delta_f}\right)\Delta_f, \tag{3}$$

where

$$M_h \stackrel{\text{def}}{=} \operatorname*{esssup}_{|\omega|\le\omega_c^h+1}|\hat{h}(\omega)|,\ m_h \stackrel{\text{def}}{=} \operatorname*{essinf}_{|\omega|\le\omega_c^h}|\hat{h}(\omega)| > 0,\ M_f \stackrel{\text{def}}{=} \operatorname*{esssup}_{\omega}|\hat{f}(\omega)|. \tag{4}$$

Proof. Let $F = f * h$. Then, from Lemma 2, we have

$$\begin{aligned}
\Delta_F^2 &= \int_{\mathbb{R}} t^2F^2(t)dt = \int_{\mathbb{R}}([tF(t)]^\wedge)^2(\omega)d\omega\\
&= 2\pi\int_{\mathbb{R}}\left([\tau f(\tau)]^\wedge(\omega)\cdot\hat{h}(\omega) + [\tau h(\tau)]^\wedge(\omega)\hat{f}(\omega)\right)^2 d\omega\\
&= 2\pi\int_{-\omega_c^f-1}^{\omega_c^f+1}\left([\tau f(\tau)]^\wedge\right)^2(\omega)\hat{h}^2(\omega)d\omega\\
&\quad + 4\pi\int_{-\omega_c^f}^{\omega_c^f}[\tau f(\tau)]^\wedge(\omega)\hat{h}(\omega)[\tau f(\tau)]^\wedge(\omega)\hat{f}(\omega)d\omega\\
&\quad + 2\pi\int_{\mathbb{R}}\left([\tau h(\tau)]^\wedge\right)^2(\omega)\hat{f}^2(\omega)d\omega.
\end{aligned} \tag{5}$$

Using Hölder inequality, the middle and the last terms of the right hand side of (5) are given by

$$
\begin{aligned}
&2\pi \int_{-\omega_c^f}^{\omega_c^f} [\tau f(\tau)]^\wedge(\omega)\hat{h}(\omega)[\tau f(\tau)]^\wedge(\omega)\hat{f}(\omega)d\omega \\
&\quad \leq 2\pi \left(\int_{-\omega_c^f-1}^{\omega_c^f+1} ([\tau f(\tau)]^\wedge)^2(\omega)\hat{h}^2(\omega)d\omega \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{-\omega_c^f}^{\omega_c^f} ([\tau f(\tau)]^\wedge)^2(\omega)\hat{f}^2(\omega)d\omega \right)^{\frac{1}{2}} \\
&\quad \leq 2\pi M_h M_f \left(\int_{-\omega_c^f-1}^{\omega_c^f+1} ([\tau f(\tau)]^\wedge)^2(\omega)d\omega \right)^{\frac{1}{2}} \\
&\qquad \times \left(\int_{-\omega_c^f}^{\omega_c^f} ([\tau f(\tau)]^\wedge)^2(\omega)d\omega \right)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
2\pi \int_{\mathbb{R}} ([\tau f(\tau)]^\wedge)^2(\omega)\hat{f}^2(\omega)d\omega \leq 2\pi M_f^2 \int_{\mathbb{R}} ([\tau f(\tau)]^\wedge)^2(\omega)d\omega
$$

respectively, where M_h, m_h and M_f are given by (4). Since

$$
\int_{\mathbb{R}} F^2(t)dt = \int_{\mathbb{R}} \hat{f}^2(\omega)\hat{h}^2(\omega)d\omega = \int_{-\omega_c^f}^{\omega_c^f} \hat{f}^2\hat{h}^2 d\omega,
$$

we obtain, from (5),

$$
\begin{aligned}
&\Delta_F^2 m_h^2 \int_{-\omega_c^f}^{\omega_c^f} \hat{f}^2(\omega)d\omega \\
&\leq 2\pi \left((\int_{-\omega_c^f-1}^{\omega_c^f+1} ([\tau f(\tau)]^\wedge)^2\hat{h}^2(\omega)d\omega)^{\frac{1}{2}} + M_f(\int_{\mathbb{R}} ([\tau h(\tau)]^\wedge)^2(\omega)d\omega)^{\frac{1}{2}} \right)^2 \\
&\leq 2\pi \left(M_h(\int_{-\omega_c^f-1}^{\omega_c^f+1} ([\tau f(\tau)]^\wedge)^2 d\omega)^{\frac{1}{2}} + M_f(\int_{\mathbb{R}} \tau^2 h^2(\tau)d\tau)^{\frac{1}{2}} \right)^2 .
\end{aligned}
\tag{6}
$$

By virtue of (4), we obtain

$$\Delta_F^2 \int_{\mathbb{R}} \hat{f}^2(\omega)d\omega \le$$

$$2\pi \left(\frac{M_h}{m_h} (\int_{\mathbb{R}} ([\tau f(\tau)]^\wedge)^2 d\omega)^{\frac{1}{2}} + \frac{M_f}{m_h} \Delta_h (\int_{\mathbb{R}} h^2(\tau)d\tau)^{\frac{1}{2}} \right)^2 .$$

Dividing both sides of (6) by $\int_{\mathbb{R}} \hat{f}^2(\omega)d\omega$, we have

$$\begin{aligned} \Delta_F^2 &\le 2\pi \left(\frac{M_h}{m_h}\Delta_f + \frac{M_f}{m_h}\Delta_h [\int_{\mathbb{R}} h^2(\tau)d\tau / \int_{\mathbb{R}} \hat{f}^2(\omega)d\omega]^{\frac{1}{2}} \right)^2 \\ &= 2\pi \left(\frac{M_h}{m_h}\Delta_f + \frac{M_f\|h\|}{\|f\|m_h}\Delta_h \right)^2 \\ &= 2\pi \left(\frac{M_h}{m_h} + \frac{M_f\|h\|\Delta_h}{m_h\|f\|\Delta_f} \right)^2 \Delta_f^2 \end{aligned} \tag{7}$$

□

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