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F-REGULAR RELATIONS

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Abstract. We define the concept of a F-regular flow as a generalization of that of a F-proximal flow, and investigate its properties.

1. Introduction

The concept of proximality has proved to be a very fruitful one for topological dynamics, giving rise to a rather extensive theory. Y.K.Kim and H.Y.Byun [4] introduced the concept of a F-proximal flow, more general than a proximal flow. In this paper, we define the concept of a F-regular flow as a generalization of that of a F-proximal flow, and investigate its properties.

2. Preliminaries

In this paper, let T be arbitrary, but fixed topological group and we consider a flow (X, T) with compact Hausdorff space X. A closed nonempty subset M of X is said to be *minimal* if the orbit xT is a dense subset of M for every $x \in M$. If X is itself minimal, we say it is a *minimal flow*. A subset A of T is said to be *syndetic* if there exists a compact subset K of T with $T = AK$. The *enveloping semigroup* $E(X)$ of (X, T) is the closure of $\{t : x \mapsto xt \mid t \in T\}$ in X^X.

A pair of points $(x, y), x, y \in X$ is *proximal* if $xp = yp$ for some $p \in E(X)$. The proximal pairs will be denoted $P(X, T)$.

We denote the endomorphisms of (X, T) by $H(X)$, and the automorphisms of (X, T) by $A(X)$. If $\phi \in H(X)$, we use the notation $\phi \in H_1(X)$

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to denote $\phi\,|_M \in H(M)$ for any minimal subset M of (X,T). Similarly, if $\phi \in A(X)$, we use the notation $\phi \in A_1(X)$ to denote $\phi\,|_M \in A(M)$ for any minimal subset M of (X,T).

A pair of points $(x,y),\ x,y \in X$ is said to be *regular* provided that $(\phi(x),y) \in P(X,T)$ for some $\phi \in H_1(X)$. The regular pairs will be denoted $R(X,T)$.

A flow (X,T) is *weakly almost periodic* (or simply w.a.p.) iff each element of $E(X)$ is continuous. It is well known that the flow (X,T) is almost periodic if and only if $E(X)$ is a compact topological group and the elements of $E(X)$ are continuous maps.

For a flow (X,T) we define the first prolongation set and the first prolongation limit set of x in X respectively, by

$$D(x) = \{y \mid\ x_it_i \to y \text{ for some } x_i \to x,\ t_i \in T\},$$
$$J(x) = \{y \mid\ x_it_i \to y \text{ for some } x_i \to x,\ t_i \to \infty\},$$

where $t_i \to \infty$ means that the net $\{t_i\}$ is ultimately outside each compact subset of T.

A point $x \in X$ is said to *have property* M if whenever there are nets $x_i \to x,\ y_i \to x$ and $\{t_i\}$ in T such that the nets $\{x_it_i\}$ is convergent, then the net $\{y_it_i\}$ is also convergent. A flow (X,T) is said to be *T-weakly equicontinuous* if $J(x,x) \subset \Delta_X$ and x has property M, for every $x \in X$.

A pair of points $(x,x'),\ x,x' \in X$ is *F-proximal* if $D(x,x') \cap \Delta \neq \emptyset$. The F-proximal pairs will be denoted $FP(X,T)$. The flow (X,T) is said to be *F-proximal* if every two points of X are F-proximal. It is checked that if (X,T) is a proximal flow, then (X,T) is a F-proximal flow.

3. F-regular relations

Definition 3.1. The relation $FR(X,T)$ is defined by the collection

$$\{(x,x') \in X \times X \mid (\phi(x),x') \in FP(X,T) \text{ for some } \phi \in H_1(X)\},$$

which is called the *F-regular relation on X.*

The flow (X,T) is said to be *F-regular* if every points of X are F-regular.

Remark 3.2. (1) $P(X,T) \subset FP(X,T) \subset FR(X,T)$.

(2) Let $X = [0,1]$ and let $\phi(x) = x^2$ for all $x \in X$. Then the map $(x,n) \mapsto \phi^n(x)$ defines an action of the integers Z on X. Indeed the flow

(X, Z) is not a proximal flow, but is a F-proximal flow (see Remark 2.3 in [4]).

LEMMA 3.3. [4] *If* (X, T) *is* T*-weakly equicontinuous, then* $FP(X, T) = P(X, T)$.

Proof. Note that if there are nets $\{x_i\}$ and $\{t_i\}$ in T such that $x_i \to x$ and $x_i t_i \to x_0$, then $x t_i \to x_0$. □

THEOREM 3.4. *If* (X, T) *is* T*-weakly equicontinuous, then* $FR(X, T) = R(X, T)$.

Proof. Let $(x, x') \in R(X, T)$. Then $(\phi(x), x') \in P(X, T)$ for some $\phi \in H_1(X)$. Since $P(X, T) \subset FP(X, T)$, it follows that $R(X, T) \subset FR(X, T)$.

To see that $FR(X, T) \subset R(X, T)$, let $(x, x') \in FR(X, T)$. Then there is a $\phi \in H_1(X)$ such that $(\phi(x), x') \in FP(X, T)$. Since (X, T) is T-weakly equicontinuous, we have $FP(X, T) = P(X, T)$ by Lemma 3.3. Therefore $(x, x') \in R(X, T)$. □

LEMMA 3.5. *(a) If* $(x, x') \in FP(X, T)$ *and* $\phi \in H(X)$, *then*

$$(\phi(x), \phi(x')) \in FP(X, T).$$

(b) If $(x, x') \in FP(X, T)$ *and* $\sigma : (X, T) \longrightarrow (Y, T)$ *is a homomorphism, then* $(\sigma(x), \sigma(x')) \in FP(Y, T)$.

Proof. (a) Let $(x, x') \in FP(X, T)$ and $\phi \in H(X)$. Then there are nets $\{x_i\}$, $\{x_i'\}$ in X and $\{t_i\}$ in T such that $x_i \to x$ and $x_i' \to x'$ and $\lim x_i t_i = \lim x_i' t_i$. The map ϕ is continuous, so $\phi(x_i) \to \phi(x)$ and $\phi(x_i') \to \phi(x')$. To complete the proof we observe that $\lim \phi(x_i) t_i = \lim \phi(x_i t_i) = \phi(\lim x_i t_i) = \phi(\lim x_i' t_i) = \lim \phi(x_i' t_i) = \lim \phi(x_i') t_i$. We thus have $(\phi(x), \phi(x')) \in FP(X, T)$.

(b) The proof is similar to that of (a). □

REMARK 3.6. In general it is not true that if $(x, x') \in FR(X, T)$ and $\phi \in H_1(X)$, then $(\phi(x), \phi(x')) \in FR(X, T)$ as the following shows. Suppose that there exists a $\psi \in H_1(X)$ such that $(\psi(x), x') \in FP(X, T)$. Then $(\phi(\psi(x)), \phi(x')) \in FP(X, T)$ by Lemma 3.5.(a). However, in general $(\phi(\psi(x)), \phi(x')) \neq (\psi(\phi(x)), \phi(x'))$.

THEOREM 3.7. *Let* $H_1(X)$ *be algebraically transitive (that is, if* $x, x' \in X$, *there is a* $\eta \in H_1(X)$ *with* $\eta(x) = x'$*) and let* $(x, x') \in FR(X, T)$ *and* $\phi \in H_1(X)$. *Then* $(\phi(x), \phi(x')) \in FR(X, T)$.

Proof. Let $(x, x') \in FR(X,T)$ and let $\phi \in H_1(X)$. There exists a $\psi \in H_1(X)$ such that $(\psi(x), x') \in FP(X,T)$. Then $(\phi(\psi(x)), \phi(x')) \in FP(X,T)$ by Lemma 3.5.(a). But since $H_1(X)$ is algebraically transitive, there is a $\eta \in H_1(X)$ with $\eta(\phi(x)) = \psi(x)$. Hence $(\phi(\eta(\phi(x))), \phi(x'))$ $= (\phi\eta(\phi(x)), \phi(x')) \in FP(X,T)$. Since $\phi\eta \in H_1(X)$, it follows that $(\phi(x), \phi(x')) \in FR(X,T)$. □

COROLLARY 3.8. *Let (X,T) be minimal and let (X,T) and $(E(X),T)$ be isomorphic. If $(x, x') \in FR(X,T)$ and $\phi \in H(X)$, then $(\phi(x), \phi(x')) \in FR(X,T)$.*

Proof. Let $(x, x') \in FR(X,T)$ and let $\phi \in H(X)$. Since (X,T) is minimal and (X,T) is isomorphic with $(E(X),T)$, we have $\phi \in H_1(X)$ and $A(X)$ is algebraically transitive by [2, Theorem 5]. It then follows from Theorem 3.7 that $(\phi(x), \phi(x')) \in FR(X,T)$. □

COROLLARY 3.9. *Let (X,T) be a w.a.p. minimal flow with T abelian. If $(x, x') \in FR(X,T)$ and $\phi \in H(X)$, then $(\phi(x), \phi(x')) \in FR(X,T)$.*

Proof. First note that if (X,T) is w.a.p. minimal, then it is almost periodic ([1, Theorem 6 of chapter 4]). Hence (X,T) is a almost periodic minimal flow with T abelian. By [3, Remark 4.6], the flows (X,T) and $(E(X),T)$ are isomorphic. □

PROPOSITION 3.10. *Let $\sigma : (X,T) \longrightarrow (Y,T)$ be an epimorphism, and assume that the only endomorphism of (X,T) is the identity. If (X,T) is F-regular, then (Y,T) is F-regular.*

Proof. For any $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$ such that $\sigma(x_1) = y_1, \sigma(x_2) = y_2$. Since (X,T) is F-regular, there exists a $\phi \in H_1(X)$ such that $(\phi(x_1), x_2) \in FP(X,T)$. Now $\phi = id_X$, so $(x_1, x_2) \in FP(X,T)$. We then have $(\sigma(x_1), \sigma(x_2)) \in FP(Y,T)$ by Lemma 3.5.(b). That is, $(y_1, y_2) \in FP(Y,T)$. Since $FP(Y,T) \subset FR(Y,T)$, we thus have (Y,T) is F-regular. □

PROPOSITION 3.11. *Let $\sigma : (X,T) \longrightarrow (Y,T)$ be an epimorphism, and assume that $H_1(Y)$ is algebraically transitive. If (X,T) is F-regular, then (Y,T) is F-regular.*

Proof. For any $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$ such that $\sigma(x_1) = y_1, \sigma(x_2) = y_2$. Since (X,T) is F-regular, there exists a $\phi \in H_1(X)$ such that $(\phi(x_1), x_2) \in FP(X,T)$. We then have $(\sigma(\phi(x_1)), \sigma(x_2)) \in FP(Y,T)$ by Lemma 3.5.(b). But since $H_1(Y)$ is algebraically transitive, there is a $\zeta \in H_1(Y)$ with $\zeta(y_1) = \sigma(\phi(x_1))$, it follows that $(y_1, y_2) \in FR(Y,T)$. We thus have (Y,T) is F-regular. □

PROPOSITION 3.12. *Let $\sigma : (X,T) \longrightarrow (Y,T)$ be an isomorphism. Then if (X,T) is F-regular, then (Y,T) is F-regular.*

Proof. For any $y_1, y_2 \in Y$, there exist $x_1, x_2 \in X$ such that $\sigma(x_1) = y_1$, $\sigma(x_2) = y_2$. Then there exists a $\phi \in H_1(X)$ such that $(\phi(x_1), x_2) \in FP(X,T)$. Applying Lemma 3.5.(b) we have $(\sigma(\phi(x_1)), \sigma(x_2)) \in FP(Y,T)$; moreover $(\sigma(\phi(\sigma^{-1}(y_1))), y_2) \in FP(Y,T)$. But since σ is a bijective map and $\phi \in H_1(X)$, it follows that $\sigma\phi\sigma^{-1} \in H_1(Y)$. Thus $(y_1, y_2) \in FR(Y,T)$. □

PROPOSITION 3.13. *Let (X,T) be a flow, and let S be a syndetic subgroup of T. Then (X,T) is F-regular if and only if (X,S) is F-regular.*

Proof. This follows immediately from the fact that $FP(X,T) = FP(X,S)$ (see Lemma 2.8 in [4]). □

REMARK 3.14. $FP(X,T)$ is a reflexive, symmetric, closed, and T-invariant relation on X, but is not in general transitive. However, $FR(X,T)$ is a reflexive and T-invariant relation on X.

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