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ON FUZZY k-IDEALS IN SEMIRINGS

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Abstract. In this paper, with the notion of fuzzy k-ideals of semirings, we discuss and review several results described in [4].

1. Introduction

L. A. Zadeh ([10]) introduced the notion of a fuzzy subset μ of a set X as a function from X into the closed unit interval $[0, 1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld ([8]). W. J. Liu ([7]) studied fuzzy ideals in rings. T. K. Dutta and B. K. Biswas ([2, 3]) studied fuzzy ideals, fuzzy prime ideals of semirings, and they defined fuzzy k-ideals and fuzzy prime k-ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k-ideals. Recently, Y. B. Jun, J. Neggers and H. S. Kim ([4]) extended the concept of an L-fuzzy (characteristic) left (resp., right) ideal of a ring to a semiring R, and showed that each level left (resp., right) ideal of an L-fuzzy left (resp., right) ideal μ of R is characteristic iff μ is L-fuzzy characteristic. The concept of semirings was introduced by H. S. Vandiver in 1935 and has since then been studied by many authors (e.g., [1,5,6,9,11]). The notion of k-ideals and Q-ideals ([1]) were applied to construct quotient semirings. In this paper, with the notion of fuzzy k-ideals of semirings, we discuss and review several results described in [4].

2. Preliminaries

An algebra $(R; +, \cdot)$ is said to be a *semiring* ([11]) if $(R; +)$ and $(R; \cdot)$

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are semigroups satisfying $a\cdot(b+c) = a\cdot b+a\cdot c$ and $(b+c)\cdot a = b\cdot a+c\cdot a$ for all $a, b, c \in R$. A semiring R may have an identity 1, defined by $1 \cdot a = a = a \cdot 1$, and a zero 0, defined by $0 + a = a = a + 0$ and $a \cdot 0 = 0 = 0 \cdot a$ for all $a \in R$.

From now on we write R and S for semirings. A non-empty subset I of R is said to be a *left* (resp., *right*) *ideal* if $x, y \in I$ and $r \in R$ imply that $x + y \in I$ and $rx \in I$ (resp., $xr \in I$). If I is both left and right ideal of R, we say I is a two-sided ideal, or simply, ideal of R. A left ideal I of a semiring R is said to be a *left k-ideal* if $a \in I$ and $x \in R$ and if $a + x \in I$ or $x + a \in I$ then $x \in I$. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.

A mapping $f : R \to S$ is said to be a *homomorphism* if $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$. We note that if $f : R \to S$ is an onto homomorphism and I is a left (resp., right) ideal of R, then $f(I)$ is a left (resp., right) ideal of S.

Definition 2.1 ([2]). A fuzzy subset μ of a semiring R is said to be a *fuzzy left* (resp.,*right*) *ideal* of R if $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(xy) \geq \mu(y)$ (resp., $\mu(xy) \geq \mu(x)$) for all $x, y \in R$. μ is a fuzzy ideal of R if it is both a fuzzy left and a fuzzy right ideal of R.

Definition 2.2 ([3]). A fuzzy ideal μ of a semiring R is said to be a *fuzzy k-ideal* of R if

$$\mu(x) \geq \min\{\max\{\mu(x + y), \mu(y + x)\}, \mu(y)\}$$

for all $x, y \in R$. If R is additively commutative then the condition reduces to $\mu(x) \geq \min\{\mu(x + y), \mu(y)\}$ for all $x, y \in R$.

Note that every fuzzy ideal of a ring is a fuzzy k-ideal.

Example 2.3 ([3]). Let μ be a fuzzy subset of the semiring N of natural numbers defined by

$$\mu(x) := \begin{cases} 0.3, & \text{if } x \text{ is odd,} \\ 0.5, & \text{if } x \text{ is non-zero even,} \\ 1, & \text{if } x = 0. \end{cases}$$

Then μ is a fuzzy k-ideal of N.

Example 2.4 ([3]). Let μ be a fuzzy subset of the semiring N of natural numbers defined by

$$\mu(x) := \begin{cases} 1, & \text{if } 7 \leq x, \\ 0.5, & \text{if } 5 \leq x < 7, \\ 0, & \text{if } 0 \leq x < 5. \end{cases}$$

Then it is easy to show that μ is a fuzzy ideal of N, but not a fuzzy k-ideal of N.

Proposition 2.5 ([3]). *Let I be a non-empty subset of a semiring R and λ_I the characteristic function of I. Then I is a k-ideal of R if and only if λ_I is a fuzzy k-ideal of R.*

3. Main Results

Y. B. Jun, J. Neggers and H. S. Kim ([4]) studied L-fuzzy ideals in semirings, and T. K. Dutta and B. K. Biswas ([3]) defined the notion of fuzzy k-ideals in semirings. With the notion of fuzzy k-ideals of semirings we discuss and review several results described in [4].

Proposition 3.1. *A fuzzy subset μ of R is a fuzzy left (resp., right) k-ideal of R if and only if, for any $t \in [0, 1]$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp., right) k-ideal of R, where $\mu_t = \{x \in R | \mu(x) \geq t\}$, which is called a level subset of μ.*

Proof. It was proved that a fuzzy subset μ is a fuzzy left (resp., right) ideal of R if and only if for any $t \in [0, 1]$ such that $\mu_t \neq \emptyset$, μ_t is a left (resp., right) ideal of R (see [4]). Assume that μ is a fuzzy k-ideal of R. Suppose that $a \in \mu_t$ and $x \in R$, and $a + x \in \mu_t$ or $x + a \in \mu_t$. Then $\mu(a) \geq t, \mu(a + x) \geq t$ or $\mu(x + a) \geq t$, and hence $\max\{\mu(a + x), \mu(x + a)\} \geq t$. Since μ is a fuzzy k-ideal of R, $\mu(x) \geq \min\{\max\{\mu(a + x), \mu(x + a)\}, \mu(a)\}$, i.e., $x \in \mu_t$. Hence μ_t is a k-ideal of R.

Conversely, assume μ_t is a k-ideal of R, for any $t \in [0, 1]$ with $\mu_t \neq \emptyset$. For any $x, a \in R$, let $\mu(a) = t_1, \mu(x + a) = t_2, \mu(a + x) = t_3$ ($t_i \in [0, 1]$). If we let $t := \min\{\max\{t_2, t_3\}, t_1\}$, then $a \in \mu_t$ and $a + x \in \mu_t$ or $x + a \in \mu_t$. Since μ_t is a k-ideal of R, we have $x \in \mu_t$, i.e.,

$\mu(x) \geq \min\{\max\{\mu(x+a), \mu(a+x)\}, \mu(a)\}$, proving that μ is a fuzzy k-ideal of R. □

Note that if μ is a fuzzy left (resp., right) k-ideal of R then the set $R_\mu := \{x \in R | \mu(x) \geq \mu(0)\}$ is a left (resp., right) k-ideal of R.

THEOREM 3.2. *Let I be any left (resp., right) k-ideal of R. Then there exists a fuzzy left (resp., right) k-ideal μ of R such that $\mu_t = I$ for some $t \in [0, 1]$.*

Proof. If we define a fuzzy subset of R by

$$\mu(x) := \begin{cases} t & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

for some $t \in [0, 1]$, then it follows that $\mu_t = I$. For a given $s \in [0, 1]$ we have

$$\mu_s = \begin{cases} \mu_0 \ (= R) & \text{if } s = 0, \\ \mu_t \ (= I) & \text{if } s \leq t, \\ \emptyset & \text{if } t < s \leq 1. \end{cases}$$

Since I and R itself are left (resp., right) k-ideals of R, it follows that every non-empty level subset μ_s of μ is a left (resp., right) k-ideal of R. By Proposition 3.1, μ is a fuzzy left (resp., right) k-ideal of R, proving the theorem. □

Let μ and ν be fuzzy subsets of R. We denote that $\mu \subseteq \nu$ if and only if $\mu(x) \leq \nu(x)$ for all $x \in X$, and $\mu \subset \nu$ if and only if $\mu \subseteq \nu$ and $\mu \neq \nu$.

THEOREM 3.3. *Let μ be a fuzzy left (resp., right) k-ideal of R. Then two level left (resp., right) k-ideals μ_s, μ_t (with $s < t$ in $[0, 1]$) of μ are equal if and only if there is no $x \in R$ such that $s \leq \mu(x) < t$.*

Proof. Suppose $s < t$ in $[0, 1]$ and $\mu_s = \mu_t$. If there exists an $x \in R$ such that $s \leq \mu(x) < t$, then μ_t is a proper subset of μ_s, a contradiction. Conversely, suppose that there is no $x \in R$ such that $s \leq \mu(x) < t$. Note that $s < t$ implies $\mu_t \subseteq \mu_s$. If $x \in \mu_s$, then $\mu(x) \geq s$, and so $\mu(x) \geq t$ because $\mu(x) \not< t$. Hence $x \in \mu_t$, and $\mu_s = \mu_t$. This completes the proof. □

Given a fuzzy k-ideal μ of R we denote by $\mathrm{Im}(\mu)$ the image set of μ.

THEOREM 3.4. *Let μ be a fuzzy left (resp., right) k-ideal of R. If $\mathrm{Im}(\mu)= \{t_1, t_2, ..., t_n\}$, where $t_1 < t_2 < ... < t_n$, then the family of left (resp., right) k-ideals μ_{t_i} $(i = 1, ..., n)$ constitutes the collection of all level left (resp., right) ideals of μ.*

Proof. If $t \in [0, 1]$ with $t < t_1$, then $\mu_{t_1} \subseteq \mu_t$. Since $\mu_{t_1} = R$, we have $\mu_t = R$ and $\mu_t = \mu_{t_1}$. If $t \in [0, 1]$ with $t_i < t < t_{i+1}$ $(1 \le i \le n-1)$, then there is no $x \in R$ such that $t \le \mu(x) < t_{i+1}$. It follows from Theorem 3.3 that $\mu_t = \mu_{t_{i+1}}$. This shows that for any $t \in L$ with $t \le \mu(0)$, the level left (resp., right) ideal μ_t is in $\{\mu_{t_i} | 1 \le i \le n\}$. This completes the proof. □

Given any two sets R and S, let μ be a fuzzy subset of R and let $f : R \to S$ be any function. We define a fuzzy subset ν on S by

$$\nu(y) := \begin{cases} \sup\limits_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, y \in S, \\ 0 & \text{otherwise,} \end{cases}$$

and we call ν the *image* of μ under f, written $f(\mu)$. For any fuzzy subset ν on $f(R)$, we define a fuzzy subset μ on R by $\mu(x) := \nu(f(x))$ for all $x \in R$, and we call μ the *preimage* of ν under f which is denoted by $f^{-1}(\nu)$.

THEOREM 3.5. *An onto homomorphic preimage of a fuzzy left (resp., right) k-ideal is a fuzzy left (resp., right) k-ideal.*

Proof. Let $f : R \to S$ be an onto homomorphism. Let ν be a fuzzy left (resp., right) k-ideal on S and let μ be the preimage of ν under f. Then it was proved that μ is a fuzzy left (resp., right) ideal of R ([4]). For any $x, y \in S$, we have

$$\begin{aligned} \mu(x) &= \nu(f(x)) \\ &\ge \min\{\max\{\nu(f(x) + f(y)),\, \nu(f(y) + f(x))\},\, \nu(f(y))\} \\ &= \min\{\max\{\nu(f(x + y)),\, \nu(f(y + x))\},\, \nu(f(y))\} \\ &= \min\{\max\{\mu(x + y),\, \mu(y + x)\}, \mu(y)\}, \end{aligned}$$

proving that μ is a fuzzy left (resp., right) k-ideal of R. □

PROPOSITION 3.6 ([4]). *Let f be a mapping from a set X to a set Y, and let μ be a fuzzy subset of X. Then for every $t \in (0,1]$,*

$$(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s}).$$

THEOREM 3.7. *Let $f : R \to S$ be an onto homomorphism and let μ be a fuzzy left (resp., right) k-ideal of R. Then the homomorphic image $f(\mu)$ of μ under f is a fuzzy left (resp., right) k-ideal of S.*

Proof. In view of Proposition 3.1 it is sufficient to show that each non-empty level subset of $f(\mu)$ is a left (resp., right) k-ideal of S. Let $(f(\mu))_t$ be a non-empty level subset of $f(\mu)$ for every $t \in [0,1]$. If $t = 0$ then $(f(\mu))_t = S$. Assume $t \neq 0$. By Proposition 3.6, $(f(\mu))_t = \bigcap_{0<s<t} f(\mu_{t-s})$. Hence $f(\mu_{t-s})$ is non-empty for each $0 < s < t$, and so μ_{t-s} is a non-empty level subset of μ for every $0 < s < t$. Since μ is a fuzzy left (resp., right) k-ideal of R, it follows from Proposition 3.1 that μ_{t-s} is a left (resp., right) k-ideal of R. Since f is an onto homomorphism, $f(\mu_{t-s})$ is a left (resp., right) k-ideal of S. Hence $(f(\mu))_t$ being an intersection of a family of left (resp., right) k-ideals is also a left (resp., right) k-ideal of S. The proof is complete. □

DEFINITION 3.8. A left (resp., right) k-ideal I of R is said to be *characteristic* if $f(I) = I$ for all $f \in \mathrm{Aut}(R)$, where $\mathrm{Aut}(R)$ is the set of all automorphisms of R. A fuzzy left (resp., right) k-ideal μ of R is said to be a *fuzzy characteristic* if $\mu(f(x)) = \mu(x)$ for all $x \in R$ and $f \in \mathrm{Aut}(R)$.

THEOREM 3.9. *Let μ be a fuzzy left (resp., right) k-ideal of R and let $f : R \to R$ be an onto homomorphism. Then the mapping $\mu^f : R \to [0,1]$, defined by $\mu^f(x) := \mu(f(x))$ for all $x \in R$, is a fuzzy left (resp., right) k-ideal of R.*

Proof. It was proved that μ^f is a fuzzy left (resp., right) k-ideal of R ([4]). For any $x, y \in R$, we have

$$\begin{aligned} \mu^f(x) &= \mu(f(x)) \\ &\geq \min\{\max\{\mu(f(x)+f(y)),\, \mu(f(y)+f(x))\},\, \mu(f(y))\} \\ &= \min\{\max\{\mu(f(x+y)),\, \mu(f(y+x))\},\, \mu(f(y))\} \\ &= \min\{\max\{\mu^f(x+y),\, \mu^f(y+x)\},\, \mu^f(x)\}, \end{aligned}$$

proving that μ^f is a fuzzy left (resp., right) k-ideal of R. □

THEOREM 3.10. *If μ is a fuzzy characteristic left (resp., right) k-ideal of R, then each level left (resp., right) k-ideal of μ is characteristic.*

Proof. Let μ be a fuzzy characteristic left (resp., right) k-ideal of R and let $f \in \mathrm{Aut}(R)$. For any $t \in [0,1]$, if $y \in f(\mu_t)$, then $\mu(y) = \mu(f(x)) = \mu(x) \geq t$ for some $x \in \mu_t$ with $y = f(x)$. It follows that $y \in \mu_t$. Conversely, if $y \in \mu_t$, then $t \leq \mu(y) = \mu(f(x)) = \mu(x)$ for some $x \in R$ with $y = f(x)$. It follows that $y \in f(\mu_t)$. This completes the proof. □

To prove the converse of Theorem 3.10, we need the following lemma.

LEMMA 3.11. *Let μ be a fuzzy left (resp., right) k-ideal of R and let $x \in R$. Then $\mu(x) = t$ if and only if $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$.*

Proof. Straightforward. □

THEOREM 3.12. *Let μ be a fuzzy left (resp., right) k-ideal of R. If each level left (resp. right) k-ideal of μ is characteristic, then μ is fuzzy characteristic.*

Proof. Let $x \in R$ and $f \in \mathrm{Aut}(R)$. If $\mu(x) = t \in [0,1]$, then by Lemma 3.11 $x \in \mu_t$ and $x \notin \mu_s$ for all $s > t$. Since each level left (resp., right) k-ideal of μ is characteristic, $f(x) \in f(\mu_t) = \mu_t$. Assume $\mu(f(x)) = s > t$. Then $f(x) \in \mu_s = f(\mu_s)$. Since f is one-to-one, it follows that $x \in \mu_s$, a contradiction. Hence $\mu(f(x)) = t = \mu(x)$, showing that μ is fuzzy characteristic. □

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