Kangweon-Kyungki Math. Jour. 8 (2000), No. 2, pp. 139–146

CONVERGENCE OF REGULARIZED SEMIGROUPS

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Abstract. In this paper, we discuss convergence theorem for contraction C-regularized semigroups. We establish the convergence of the sequence of generators of contraction regularized semigroups in some sense implies the convergence of the sequence of the corresponding contraction regularized semigroups. Under the assumption that $R(C)$ is dense, we show the convergence of generators is implied by the convergence of C-resolvents of generators.

1. Introduction

Let X be a Banach space and let A be a linear operator from $D(A) \subseteq X$ into X. The abstract Cauchy problem for A with initial data $x \in X$ consists of finding a solution to

$$\frac{d}{dt}u(t) = Au(t), \quad t \geq 0, \quad u(0) = x.$$

From the theory of C_0 semigroup, if A is the infinitesimal generator of a C_0 semigroup, then the abstract Cauchy problem for A has a solution and the solution is given by the C_0 semigroup. And it is well known that a densely defined linear operator with nonempty resolvent set is the infinitesimal generator of a C_0 semigroup. When A does not generate a C_0 semigroup, the C_0 semigroups cannot be applied directly to the abstract Cauchy problem for A.

For the abstract Cauchy problem for a non densely defined linear operator A, regularized semigroups were independently introduced by Da Prato [2], and by Davies and Pang [3], where they were called C-semigroups. There are some problems for the terminology C - semigroup. In the terminology of C-semigroups, it is not clear that

Received July 5, 2000.
1991 Mathematics Subject Classification: 47H05.
Key words and phrases: Cauchy Problem, regularized semigroups, generators, solution space, convergence.

a specific C is given. It may imply C-regularized semigroup for some C instead of C-regularized semigroup for this specific C.

The class of operators that generate regularized semigroups is much larger than the class of generators of C_0 semigroups. So regularized semigroups are generalizations of C_0 semigroups which can be applied to the abstract Cauchy problem where the C_0 semigroups cannot be applied directly. Examples of this are the backwards heat equation, the Schrödinger equation on L^p, $p \neq 2$ and the Cauchy problem for the Laplace equation. A non densely defined linear operator A may be a generator of regularized semigroup and in this case the solution of the abstract Cauchy problem is given by regularized semigroup.

It is known [8] that a C_0 semigroup depends continuously on its infinitesimal generator and the infinitesimal generator also depends continuously on the corresponding C_0 semigroup. In this paper, we will establish similar result for contraction regularized semigroups. That is, the solution of the abstract Cauchy problem given by regularized semigroups depends on A. The idea we use in this paper is to use the Hille-Yosida space and then to apply several inequalities appeared in nonlinear semigroup theory in [1]. Under the assumption that $R(C)$ is dense in X, we show that the convergence of generators implies the convergence of C-resolvent of generators (cf. [6]).

2. Preliminaries

Throughout this paper, X will be a Banach space. C will be a bounded injective linear operator on X. For an operator A, we will write $D(A)$ for its domain and $R(A)$ for its range.

We start with the definitions and properties of regularized semigroups.

DEFINITION 2.1. The strongly continuous family $\{S(t) : t \geq 0\}$ of bounded linear operators on X is said to be a C-regularized semigroup if

(1) $S(0) = C$,
(2) $S(t+s)C = S(t)S(s)$ for $t,\ s \geq 0$.

A C-regularized semigroup $\{S(t) : t \geq 0\}$ is exponentially bounded if there exist $M \geq 0$ and $\omega \in R$ such that $||S(t)|| \leq Me^{\omega t}$ for $t \geq 0$.

The linear operator A is a generator of a C-regularized semigroup $\{S(t) : t \geq 0\}$ if

$$Ax = C^{-1}\left(\lim_{t\to 0}\frac{1}{t}(S(t)x - Cx)\right)$$

with

$$D(A) = \{x \in X : \lim_{t\to 0}\frac{1}{t}(S(t)x - Cx) \text{ exists and is in } R(C)\}.$$

When we do not want to specify C, we will say regularized semigroups. The existence of generator of regularized semigroup is the important information about the existence of nontrivial solutions of the abstract Cauchy problem. Moreover, the choice of C tells us how many solutions we expect; the larger the range of C, the more solutions we have (see[4]).

DEFINITION 2.2. The complex number λ is in $\rho_C(A)$, the C-resolvent set of A, if $\lambda - A$ is injective and $R(C) \subset R(\lambda - A)$.

Next, we present some known facts about C-regularized semigroup and its generator, which will be used in the sequel (see [7]).

LEMMA 2.3. *Suppose that $\{S(t) : t \geq 0\}$ is an exponentially bounded C-regularized semigroup generated by A. Then*

(1) *A is closed and $R(C) \subseteq \overline{D(A)}$.*

(2) *$(\omega, \infty) \subseteq \rho_C(A)$. For every $\lambda > \omega$ and $n \in N$,*

$$R(C) \subset R((\lambda - A)^n)$$

and

$$(\lambda - A)^{-n}C = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-\lambda t}S(t)dt.$$

(3) *For all $x \in X$, $A(\int_0^t S(s)xds) = S(t)x - Cx$. That is, $S(t)x$ is a mild solution of*

$$\frac{d}{dt}u(t) = Au(t), \quad u(0) = Cx.$$

Next, we will introduce the solution space that contains all initial data for which the abstract Cauchy problem has a unique solution, on which A generates a C_0 semigroups.

DEFINITION 2.4. Suppose A as no eigenvalues in $(0, \infty)$. The exponentially bounded solution space for A is the Banach space defined by

$$Z_\omega = \{x \in X : \text{ The abstract Cauchy problem has a mild solution } u(t,\, x) \text{ such that } e^{-\omega t}u(t,\, x) \text{ is bounded and uniformly continuous}\}$$

with $||x||_{Z_\omega} = \sup\{e^{-\omega t}||u(t,\, x)|| : t \geq 0\}$ for $x \in Z_\omega$.

The space Z_0 is called the Hille-Yosida space for A

The weak Hille-Yosida space for A is the Banach space defined by

$$Y = \{x \in X : x \in R((\lambda - A)^n) \text{ for } \lambda > 0 \text{ and } n \in N \text{ with}$$
$$||x||_Y = \sup\{\lambda^n||(\lambda - A)^{-n}x|| : \lambda > 0,\, n = 0,\, 1,\, \cdots\} < \infty\}.$$

We have the following relation between Y and Z_0 (see [5]).

LEMMA 2.5. *Suppose A has no eigenvalue in $(0, \infty)$. Then*

(1) $Z_0 \subseteq Y$ *and* $||x||_{Z_0} = ||x||_Y$ *for all* $x \in X$.
(2) $A|_{Z_0}$ *generates a contraction* C_0 *semigroup on* Z_0.

3. Convergence of Contraction C-Regularized Semigroups

A C-regularized semigroup $\{S(t) : t \geq 0\}$ is of contraction if $||S(t)x|| \leq ||Cx||$ for $t \geq 0$ and $x \in X$.

Let A be the generator of a contraction C-regularized semigroup and let Z_0 be the Hille-Yosida space for A. Then $||x|| \leq ||x||_Y$ and $||Cx||_{Z_0} \leq ||Cx||$, by Lemma 2.3.

THEOREM 3.1. *Let $\{S(t) : t \geq 0\}$ and $\{W(t) : t \geq 0\}$ be C-regularized semigroups generated by A satisfying $||S(t)|| \leq Me^{\omega t}$ and $||W(t)|| \leq Me^{\omega t}$ for all $t \geq 0$. Then $S(t) = W(t)$ for all $t \geq 0$.*

Proof. Let $x \in X$ and $\lambda > \omega$. Then

$$(\lambda - A)\int_0^\infty e^{-\lambda t}S(t)xdt = Cx \quad\text{and}\quad (\lambda - A)\int_0^\infty e^{-\lambda t}W(t)xdt = Cx.$$

Since $(\lambda - A)$ is injective, $\int_0^\infty e^{-\lambda t}(S(t) - W(t))xdt = 0$. Therefore $S(t)x = T(t)x$ for all $t \geq 0$. □

By Theorem 3.3 in [7] and Theorem 5.16 in [5], we obtain the following result.

THEOREM 3.2. *Let $\{S(t) : t \geq 0\}$ be a contraction C-regularized semigroup generated by A. Then $R(C) \subseteq Z_0$ and $S(t)x = T(t)Cx$ for all $x \in X$, where $T(t)$ is a contraction C_0 semigroup generated by $A|_{Z_0}$ with $D(A|_{Z_0}) = \{x \in D(A) \cap Z_0 : Ax \in Z_0\}$ on Z_0.*

By the above theorems and the exponential formula, we obtain

$$S(t)x = \lim_{n\to\infty} \left(I - \frac{t}{n}A\right)^{-n} Cx \quad \text{for all } x \in X.$$

THEOREM 3.3. *Let $\{S_n(t) : t \geq 0\}$ be a contraction C-regularized semigroup generated by A_n. Let $\{S(t) : t \geq 0\}$ be a contraction C-regularized semigroup generated by A. Suppose that $D(A) \subseteq D(A_n)$, $(0, \infty) \subseteq \rho(A_n)$ for all $n \in N$ and for $\lambda > 0$*

$$\lim_{n\to\infty} (I - rA_n)^{-1}x_n = (I - rA)^{-1}x,$$

where $x \in R(I - rA)$ and $\lim_{n\to\infty} x_n = x$.
Suppose that $C(D(A))$ is dense in $R(C)$. Then

$$\lim_{n\to\infty} S_n(t)x = S(t)x \quad \text{for all } x \in X,$$

and the convergence is uniform on bounded t-intervals.

Proof. Let $Z_0^{(n)}$ and Z_0 be the Hille-Yosida spaces for A_n and A, respectively. Let $x \in D(A)$ and let $0 \leq t \leq T$. Then

$$\begin{aligned} S_n(t)x - S(t)x &= T_n(t)Cx - T(t)Cx \\ &= T_n(t)Cx - T_n(t)C(I - rA_n)^{-1}x \\ &\quad + T_n(t)C(I - rA_n)^{-1}x - (I - t/kA_n)^{-k}(I - rA_n)^{-1}Cx \\ &\quad + (I - t/kA_n)^{-k}(I - rA_n)^{-1}Cx - (I - t/kA_n)^{-k}Cx \\ &\quad + (I - t/kA_n)^{-k}Cx - (I - t/kA)^{-k}Cx \\ &\quad + (I - t/kA)^{-k}Cx - T(t)Cx. \end{aligned}$$

Since $rA_n(I - rA_n)^{-1}Cx = (I - rA_n)^{-1}Cx - Cx$, there exists N_1 such that for all $n \geq N_1$

$$||A_n(I - rA_n)^{-1}Cx|| \leq ||A(I - rA)^{-1}Cx|| + 1 \leq ||ACx|| + 1.$$

Since $\{T_n(t) : t \geq 0\}$ is a contraction C_0 semigroup on $Z_0^{(n)}$, we have

$$\begin{aligned}||T_n(t)Cx &- T_n(t)C(I-rA_n)^{-1}x|| \\ &\leq ||T_n(t)Cx - T_n(t)C(I-rA_n)^{-1}x||_{Z_0^{(n)}} \\ &\leq ||Cx - C(I-rA_n)^{-1}x||_{Z_0^{(n)}} \leq ||Cx - C(I-rA_n)^{-1}x|| \\ &\leq r||A_n(I-rA_n)^{-1}Cx|| \leq r(||ACx||+1),\end{aligned}$$

for all $n \geq N_1$.

In the proof of Theorem I in [1] we have the following inequalities

$$\begin{aligned}||T_n(t)(I&-rA_n)^{-1}Cx - (I-t/kA_n)^{-k}(I-rA_n)^{-1}Cx|| \\ &\leq ||T_n(t)(I-rA_n)^{-1}Cx - (I-t/kA_n)^{-k}(I-rA_n)^{-1}Cx||_{Z_0^{(n)}} \\ &\leq 2t\sqrt{1/k}||A_n(I-rA_n)^{-1}Cx||_{Z_0^{(n)}} \\ &\leq 2t\sqrt{1/k}||A_n(I-rA_n)^{-1}Cx|| \\ &\leq 2t\sqrt{1/k}(||ACx||+1),\end{aligned}$$

for all $n \geq N_1$.

By Lemma 2.3, we have

$$\begin{aligned}||(I-t/kA_n)^{-k}&(I-rA_n)^{-1}Cx - (I-t/kA_n)^{-k}Cx|| \\ &= ||(I-t/kA_n)^{-k}C(I-rA_n)^{-1}x - (I-t/kA_n)^{-k}Cx|| \\ &\leq ||C(I-rA_n)^{-1}x - Cx|| = r||A_n(I-rA_n)^{-1}Cx|| \\ &\leq r(||ACx||+1),\end{aligned}$$

for all $n \geq N_1$.

Finally, we have the following inequalities

$$\begin{aligned}||(I-t/kA)^{-k}Cx &- T(t)Cx|| \\ &\leq ||(I-t/kA)^{-k}Cx - T(t)Cx||_{Z_0} \\ &\leq 2t\sqrt{1/k}||ACx||_{Z_0} \leq 2t\sqrt{1/k}||ACx||.\end{aligned}$$

Therefore we have for all $n \geq N_1$,

$$\begin{aligned}||S_n(t)x &- S(t)x|| \\ &\leq r(||ACx||+1) + 2t\sqrt{1/k}(||ACx||+1) + r(||ACx||+1) \\ &\quad + ||(I-t/kA_n)^{-k}Cx - (I-t/kA)^{-k}Cx|| + 2t\sqrt{1/k}||ACx||.\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose $r_0 > 0$ such that for all $0 < r < r_0$

$$2r(||ACx|| + 1) \leq \frac{\varepsilon}{3}.$$

Then choose k such that

$$4T\sqrt{\frac{1}{k}}(||ACx|| + 1) \leq \frac{\varepsilon}{3}.$$

By hypothesis, there exists N_2 such that for all $n \geq N_2$

$$||(I - t/kA_n)^{-k}Cx - (I - t/kA)^{-k}Cx|| \leq \frac{\varepsilon}{3}.$$

Thus $||S_n(t)x - S(t)x|| \leq \varepsilon$ for all $x \in D(A)$ and $n \geq \max\{N_1, N_2\}$.

By hypothesis, for $x \in X$ there exists $x_k \in D(A)$ such that

$$\lim_{k\to\infty} Cx_k = Cx.$$

$$\begin{aligned}
&||S_n(t)x - S(t)x|| \\
&\quad \leq ||S(t)x - S_n(t)x_k|| + ||S_n(t)x_k - S(t)x_k|| + ||S(t)x_k - S(t)x|| \\
&\quad \leq ||Cx - Cx_k|| + ||S_n(t)x_k - S(t)x_k|| + ||Cx - Cx_k||.
\end{aligned}$$

Therefore the result follows. □

We examine the relation between the convergence of generators A_n and the convergence of $(I - rA_n)^{-1}$.

REMARK. Suppose that $R(C)$ is dense in X. By Lemma 4.2 in [7], $(0, \infty) \subseteq \rho(A)$, $(0, \infty) \subseteq \rho(A_n)$ for all n and $||(I - rA_n)^{-1}x|| \leq ||x||$ for all $x \in X$. Assume that $\lim A_n \supseteq A$, that is for $x \in D(A)$ there exist $z_n \in D(A_n)$ such that $z_n \to x$, $A_nz_n \to Ax$ as $n \to \infty$. Let $y_n = (I - rA_n)^{-1}$ and $y = (I - rA)^{-1}x$, as $n \to \infty$. And

$$||(I - rA_n)(y_n - z_n)|| \geq ||(I - rA_n)^{-1}(I - rA_n)(y_n - z_n)|| = ||y_n - z_n||.$$

Therefore we have $\lim_{n\to\infty}(I - rA_n)^{-1}x_n = (I - rA)^{-1}x$.

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