Kangweon-Kyungki Math. Jour. **8** (2000), No. 2, pp. 121–126

ASYMPTOTIC PROPERTIES OF NONEXPANSIVE SEQUENCES IN BANACH SPACES

Jong An Park and Yang Seob Park

Abstract. B.Djafari Rouhani and W.A.Kirk [3] proved the following theorem:
Let Xbe a reflexive Banach space and $(x_n)_{n\geq 0}$ be a nonexpansive (resp., firmly nonexpansive)sequence in X. Then the set of weak $\omega-$limit points of the sequence $(\frac{x_n}{n})_{n\geq 1}$(resp., $(x_{n+1}-x_n)_{n\geq 0}$) always lies on a *convex* subset of a sphere centered at the origin of radius $d=\lim_{n\to\infty}\frac{\|x_n\|}{n}$.
In this paper we show that the above theorem for nonexpansive(resp., firmly nonexpansive) sequences holds in a general Banach space(resp., a strictly convex dual X^*).

1. Introduction

Let X be a real Banach space; the norm of both X and its dual X^* are denoted by $\|\|$; we denote strong convergence and weak convergence in X respectively by $\to$ and $\rightharpoonup$. The duality map J from X into the family of nonempty closed convex subsets of X^* is defined by $J(x)=\{x^*\in X^*:(x,x^*)=\|x\|^2=\|x^*\|^2\}$. We say that a sequence $(x_n)_{n\geq 0}$ is *nonexpansive* (resp.,*firmly nonexpansive*) if $\|x_{i+1}-x_{j+1}\|\leq\|x_i-x_j\|$ for all $i,j\geq 0$(resp., if the function $f:[0,1]\to[0,\infty)$ defined by $f(t)=\|(1-t)(x_i-x_j)+t(x_{i+1}-x_{j+1})\|$ is nonincreasing for all $i,j\geq 0$). Firmly nonexpansive sequences are also characterized by the inequality

$$\|x_{i+1}-x_{j+1}\|\leq\|(1-t)(x_i-x_j)+t(x_{i+1}-x_{j+1})\|$$

for all $i,j\geq 0, t\in[0,1]$.

Received May 30, 2000.
1991 Mathematics Subject Classification: Primary 47H09.
Key words and phrases: asymptotic behavior, nonexpansive and firmly nonexpansive sequences, weak convergence,strictly convex.

In 1971 A.Pazy proved that if T is nonexpansive in a Hilbert space, then sequence $\{T^n x/n\}$ always converges strongly. Since then the asymptotic behavior of nonexpansive mappings has been extended to a more general space and to the firmly nonexpansive mappings and to nonexpansive semigroups.(See, e.g.,[4, 5]).

In this context B.Djafari Rouhani [1, 2] defined nonexpansive sequences and firmly nonexpansive sequences and studied their asymptotic behaviors.

The present paper concerns about Proposition 1 and Theorem 1,2 in [3] without reflexivity.

2. Asymptotic Behavior

We follow the notations in [3],

$$K_n = \overline{conv}\{(x_{i+1} - x_i)\}_{i \geq n}$$

$$K = \bigcap_{n=0}^{\infty} K_n$$

$$F_n = \overline{conv}\{(\frac{x_k - x_0}{k})\}_{k \geq n}$$

$$F = \bigcap_{n=0}^{\infty} F_n$$

$$S_d = \{x \in X : \|x\| = d\}.$$

Remark 1 in [3] states that if $H_n := \overline{conv}\{(\frac{x_k}{k})\}_{k \geq n}$ and $H = \cap_{n=1}^{\infty} H_n$, then $H = F$.

In analogy of Lemma 4 in [5] we can obtain the following lemma.

LEMMA 1. *Let $(x_n)_{n \geq 0}$ be a nonexpansive sequences in a Banach space X. And let $d = \lim_{n \to \infty} \|\frac{x_n}{n}\|$. Then there exists $x^* \in X^*$ such that*

$$(x^*, \frac{x_m - x_0}{m}) \geq \|x^*\|^2 = d^2$$

for all $m \geq 1$.

Proof. The proof is similar to that of Lemma 4 in [5]. So it is omitted. □

THEOREM 1. *Let X be a Banach space,$(x_n)_{n\geq 0}$ a nonexpansive sequence in X, and $d = \lim_{n\to\infty} \|\frac{x_n}{n}\|$. Then*

$$\omega_w(\{\frac{x_n}{n}\}) \subseteq F \cap S_d.$$

Proof. Suppose that $\{\frac{x_{n_k}}{n_k}\}$ converges weakly to $z \in \omega_w(\{\frac{x_n}{n}\})$. Then

$$\|z\| \leq \liminf_{n_k\to\infty} \|\frac{x_{n_k}}{n_k}\| = d$$

and by Lemma 1,

$$(x^*, z) = \lim_{n_k\to\infty} (x^*, \frac{x_{n_k}}{n_k}) \geq d^2.$$

So $\|z\| \geq d$ and $z \in S_d$.

Since H_n is closed and convex, it is weakly closed and $z \in H_n$ for all $n \geq 1$. Therefore $\|z\| \in \cap_{n=1}^{\infty} H_n = H = F$. □

In [3] B.Djafari Rouhani and W.A. Kirk characterized the sets $F \cap S_d$, $K \cap S_d$. Without reflexivity of X, we prove the following theorem.

THEOREM 2. *Let $(x_n)_{n\geq 0}$ be a nonexpansive sequence in a Banach space X. Then $F_n \cap S_d$ is convex for all $n \geq 1$. In Particular $F \cap S_d$ is convex.*

Proof. It is sufficient to prove that for all $x, y \in F_n \cap S_d, 0 \leq \lambda \leq 1$,

$$\|(1-\lambda)x + \lambda y\| = d.$$

By Lemma 1, for all $w \in F_n$

$$(x^*, w) \geq d^2.$$

So

$$(x^*, (1-\lambda)x + \lambda y) \geq d^2.$$

Hence

$$\|(1-\lambda)x + \lambda y\| \geq d.$$

On the other hand, since $x, y \in S_d$

$$\|(1-\lambda)x + \lambda y\| \leq d.$$

Therefore

$$\|(1-\lambda)x + \lambda y\| = d.$$

□

Being hinted by Corollary 2 of A.T. Plant and S. Reich [5], we obtain the following lemma which could be compared with Theorem 3,1 of [3]. We need Lemma 5 in [5] which characterizes the strict convexity as the duality map as follows:

A Banach space X is strictly convex iff its duality map is injective in the sense that $J(x) \cap J(y) \neq \emptyset$ implies $x = y$.

LEMMA 2. *If $(x_n)_{n\geq 0}$ is a nonexpansive sequence in a Banach space X with strictly convex dual X^* and let $d = \lim_{n\to\infty} \|\frac{x_n}{n}\|$, then there exists $z \in X^*$ such that*

$$(z, \frac{x_{m+i} - x_i}{m}) \geq \|z\|^2 = d^2$$

for all $m \geq 1, i \geq 0$.

Proof. Since $\{\frac{x_n}{n}\}$ is bounded in X which is identified with its natural injection in X^{**}, let us choose one weak-star subsequential limit a of $\{\frac{x_n}{n}\}$. So

$$\|a\| \leq \liminf_{n\to\infty} \|\frac{x_n}{n}\| = d$$

And for any $i \geq 0$, the subsequence $(x_{n+i})_{n\geq 0}$ is also a nonexpansive sequence. So by Lemma 1 there exists $z(i) \in X^*$ such that

$$(z(i), \frac{x_{m+i} - x_i}{m}) \geq \|z(i)\|^2 = d^2 \text{............}(*)$$

for all $m \geq 1$. Here

$$\lim_{m\to\infty} \|\frac{x_{m+i}}{m}\| = \lim_{m\to\infty} \|(\frac{m+i}{m})\frac{x_{m+i}}{m+i}\| = \lim_{n\to\infty} \|\frac{x_n}{n}\| = d.$$

Since some subsequence of $\{\frac{x_{m+i}}{m} = (\frac{m+i}{m})\frac{x_{m+i}}{m+i}\}$ converges weak-star to $a \in X^{**}$, in (*) we take the subsequential limit. Then

$$(z(i), a) \geq \|z(i)\|^2 = d^2.$$

And

$$d^2 \geq \|z(i)\|\|a\| \geq (z(i), a) \geq \|z(i)\|^2 = d^2.$$

Therefore

$$(z(i), a) = \|a\|^2 = \|z(i)\|^2.$$

for all $i \geq 0$. So for the usual duality map J^* on X^{**},

$$a \in J^*(z(i)) \cap J^*(z(j)).$$

for all $i, j \geq 0$. Since $X^*$is strictly convex, $z \equiv z(i)$ for all $i \geq 0$. Hence

$$(z, \frac{x_{m+i} - x_i}{m}) \geq \|z\|^2 = d^2$$

for all $m \geq 1, i \geq 0$. □

It is known in [2, 6] that for a firmly nonexpansive sequence $(x_n)_{n \geq 0}$,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = d = \lim_{n \to \infty} \|\frac{x_n}{n}\|.$$

THEOREM 3. *Let X be a Banach space with a strictly convex dual X^*, $(x_n)_{n \geq 0}$ a firmly nonexpansive sequence in X, and $d = \lim_{n \to \infty} \|\frac{x_n}{n}\| = \lim_{n \to \infty} \|x_{n+1} - x_n\|$. Then*

$$\omega_w(\{x_{n+1} - x_n\}) \subseteq K \cap S_d.$$

Proof. Since a firmly nonexpansive sequence is a nonexpansive sequence, there exists $z \in X^*$ such that

$$(z, x_{i+1} - x_i) \geq \|z\|^2 = d^2$$

for all $i \geq 0$. by Lemma 2 for $m = 1$. So by the definition of K_n,

$$(z, w) \geq \|z\|^2 = d^2$$

for all $w \in K_n, n \geq 0$. If $x_{n_k+1} - x_{n_k}$ converges weakly to $w' \in X$, then

$$\|w'\| \leq \liminf_{n \to \infty} \|x_{n_k+1} - x_{n_k}\| = d$$

on the other hand since K_n is weakly closed, $w' \in K_n$ for all $n \geq 0$ i.e., $w' \in \cap_{n=0}^{\infty} K_n = K$ and $(z, w') \geq \|z\|^2 = d^2$. So $\|w'\| \leq d$. Therefore $w' \in K \cap S_d$. □

THEOREM 4. *Let $(x_n)_{n \geq 0}$ be a firmly nonexpansive sequence in a Banach space X with a strictly convex dual X^*. Then $K_n \cap S_d$ is convex for all $n \geq 1$. In Particular $K \cap S_d$ is convex.*

Proof. The proof depends on Lemma 2 for $m = 1$. And its proof is similar to that of Theorem 2. So we omit it. □

References

[1] B.Djafari Rouhani, *Asymptotic behavior of unbounded nonexpansive sequences in Banach spaces*, Proc. Amer. Math. Soc. 117(1993),951-956.

[2] B.Djafari Rouhani, *Asymptotic behavior of firmly nonexpansive sequences* , Proc. Amer. Math. Soc. 123(1995), 771-777.

[3] B.Djafari Rouhani and W.A.Kirk, *Asymptotic properties of nonexpansive iterations in reflexive spaces*, J. Math. Anal. Appl. 236(1999),281-289.
[4] E. Kohlberg and A. Neyman, *Asymptotic behavior of nonexpansive mappings in normed linear spaces* , Israel J. Math. 38(1981), 269-275.
[5] A. Plant and S. Reich, *The asymptotics of nonexpansive iterations*, J. Funct. Anal. Appl. 54(1983),308-319.
[6] S.Reich and I.Shafrir, *The asymptotic behavior of firmly nonexpansive mappings*, Proc. Amer. Math. Soc. 101(1987), 246-250.

Department of Mathematics
Kangwon National University
Chunchon
Kangwondo, 200–701
KOREA
E-mail: `jongan@kangwon.ac.kr`