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REMARKS ON GAUSSIAN OPERATOR SEMI-STABLE DISTRIBUTIONS

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Abstract. For a linear operator Q from R^d into R^d, $\alpha > 0$ and $0 < b < 1$, the Gaussian (Q, b, α)-semi-stability of probability measures on R^d is investigated.

1. Introduction

Let R^d be the d-dimensional Euclidean space. In [2], one of the authors obtained the complex characterization of Gaussian operator semi-stable distribution on R^d. In this paper, we consider a description for Gaussian operator semi-stable distribution in a real form.

We will use some notation. $\mathcal{B}(R^d)$ is the collection of Borel sets in R^d. $\mathcal{P}(R^d)$ is the collection of probability measures (distributions) defined on $\mathcal{B}(R^d)$. $\widehat{\mu}(z)$, $z \in R^d$ is the characteristic function of $\mu \in \mathcal{P}(R^d)$. $End(R^d)$ is the set of linear operators from R^d into R^d. The identity operator is denoted by I. We denote Euclidean norm of x by $|x|$ and Euclidean inner product of x and y by $\langle x, y\rangle$. For $B \in End(R^d)$ and $r > 0$, we define $r^B = \exp\{B \log r\} = \sum(B \log r)^n/n!$. Let $I(R^d)$ be the collection of infinitely divisible distributions defined on $\mathcal{B}(R^d)$. We denote the b-th convolution power of $\mu \in I(R^d)$ by μ^b. The class of linear operators on R^d whose eigenvalues have positive real parts is denoted by $M_+(R^d)$.

Fix $\alpha > 0$ and $Q \in M_+(R^d)$. A distribution $\mu \in \mathcal{P}(R^d)$ is called (Q, b, α)-*semi-stable* if $\mu \in I(R^d)$ and there are a number $b \in (0, 1)$ and a vector $c(b) \in R^d$ such that

$$\mu^{b^\alpha} = b^Q \mu * \delta_{c(b)}.$$

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Here $\delta_{c(b)}$ is the delta distribution at $c(b)$. It is called *strictly* (Q, b, α)-*semi-stable* if there is $b \in (0,1)$ such that

$$\mu^{b^\alpha} = b^Q \mu.$$

The relationship between the class of (Q, b, α)-semi-stable distributions and that of strictly (Q, b, α)-semi-stable distributions has been discussed in [3]. Note that the class of (Q, b, α)-semi-stable Gaussian distributions on R^d is a proper subclass of Gaussian operator semi-stable distributions on R^d. See [7].

Our results of this paper are generalizations of some results in [8,9] to operator semi-stable case. These results are got by using the same manners as are done for operator stable distributions in [8].

Preliminary results are given in Section 2 and we write our results in Section 3. Proofs are given in Section 4.

2. Preliminaries

Let $B = b^Q$. For $x \in C^d$, $\overline{x}$ stands for the complex conjugate of x, that is, each component of $\overline{x}$ is the complex conjugate of the corresponding component of x. Let $\vartheta_1, \ldots, \vartheta_{q+2r}, (1 \le j \le q+2r)$ be all distinct eigenvalues of B, where q is the number of distinct real eigenvalues and $2r$ is the number of distinct non-real eigenvalues of $B(q \ge 0, 2r \ge 0)$. Let $\vartheta_j = \sigma_j + i\rho_j$ with σ_j and ρ_j being real. Let $f(\xi)$ be the minimal polynomial of B such that

$$f(\xi) = (\xi - \vartheta)^{m_1} \ldots (\xi - \vartheta_{q+2r})^{m_{q+2r}}.$$

Then

$$f(\xi) = f_1(\xi)^{m_1} \ldots f_{q+r}(\xi)^{m_{q+r}},$$

where

$$f_j(\xi) = \begin{cases} \xi - \vartheta_j & \text{for} \quad 1 \le j \le q \\ (\xi - \sigma_j)^2 + \rho_j{}^2 & \text{for} \quad q+1 \le j \le q+r. \end{cases}$$

We write W_j for the kernel of $f_j(B)^{m_j}$ in R^d, $1 \le j \le q+r$. The projector onto W_j in the direct sum decomposition

$$R^d = W_1 \oplus \cdots \oplus W_{q+r}$$

is written by U_j. Denote

$$V_j = \text{Kernel}(B - \vartheta_j)^{m_j} \quad \text{in} \quad C^d, \quad 1 \le j \le q + 2r.$$

Then we have that

$$C^d = V_1 \oplus \cdots \oplus V_{q+2r}. \tag{2.1}$$

Let T_j be the projector onto V_j in the decomposition (2.1). We denote the adjoint of a linear operator T by T'. Set

$$W_j' = \text{Kernel} f_j(B')^{m_j} \quad \text{in} \quad R^d \quad \text{for} \quad 1 \le j \le q + r$$

and

$$V_j' = \text{Kernel}(B' - \overline{\vartheta_j})^{m_j} \quad \text{in} \quad C^d \quad \text{for} \quad 1 \le j \le q + 2r.$$

Then we have

$$C^d = V_1' \oplus \cdots \oplus V_{q+2r}' \tag{2.2}$$

and

$$R^d = W_1' \oplus \cdots \oplus W_{q+r}'. \tag{2.3}$$

We see that V_j' and V_k are orthogonal for $j \neq k$ and T_j' is the projector of C^d onto V_j' in the decomposition (2.2).

We set

$$\Lambda = \{j : 1 \le j \le q + r \quad \text{satisfying} \quad |\vartheta_j| < b^{1/2}\}.$$

Let $W_\Lambda = \oplus_{j\in\Lambda} W_j$, let $S_\Lambda = \{x \in W_\Lambda : | x |_Q \le 1 \ \text{ and } | B^{-1}x |_Q > 1\}$ and $\mathcal{B}(S_\Lambda)$ as the class of Borel set in S_Λ. Here $| \cdot |_Q$ is the norm such that

$$| x |_Q = \int_0^1 \frac{| u^Q x |}{u} du, \quad x \in R^d.$$

The reason that we use the norm $| \cdot |_Q$ is given in [7]. It is well-known that $\mu \in I(R^d)$ is infinitely divisible if and only if $\widehat{\mu}(z)$ has form

$$\widehat{\mu}(z) = \exp\left\{ i\langle \gamma, z\rangle - \frac{1}{2}\langle Az, z\rangle + \int_{R^d} G(z, x)\nu(dx) \right\},$$

where $G(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}$, γ is a vector in R^d, A is a symmetric nonnegative definite operator and ν is a measure (called Lévy measure μ) on R^d satisfying $\nu(\{0\}) = 0$ and

$$\int |x|^2(1+|x|^2)^{-1}\nu(dx) < \infty.$$

This representation is unique and called the Lévy representation (γ, A,ν). We call μ a purely non-Gaussian in the case of $A = 0$. If $\gamma = 0$ and $A = 0$, then we call μ a centered purely non-Gaussian. If $\gamma = 0$ and $\nu = 0$, then μ is called a centered Gaussian. In [4], we note that recurrence and transience of Lévy process on R^2 associated with (Q,b,α)-semi-stable distribution depend on whether it is centered Gaussian or not. For a real symmetric nonnegative definite operator A, $\phi_A(z)$ stands for $\langle Az, z\rangle$. Here $\langle\ \ \rangle$ denotes the Hermitian inner product on C^d. Let $J = \{j : 1 \le j \le q+2r, |\vartheta_j|^2 = b^\alpha\}$. The following proposition is obvious from the results of Łuczak in [5,6] and Theorem 3.1 in [2].

PROPOSITION 2.1. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let $\mu \in I(R^d)$ with the Lévy representation (γ, A, ν). Then μ is (Q,b,α)-semi-stable if and only if*

$$AT_j' = 0 \quad \text{for all} \quad j \notin J, \tag{2.4}$$

$$(B-\vartheta_j)AT_j' = 0 \quad \text{for all} \quad j \in J, \tag{2.5}$$

$$\nu(E) = \int_{S_\Lambda} \sum_{n=-\infty}^{n=\infty} b^n I_E(B^{-n}x)\nu_0(dx), \quad E \in \mathcal{B}(R^d),$$

where ν_0 is a finite Borel measure on S_Λ.

3. Results

Denote $N_j = \mathrm{Ker} f_j(B)$ in R^d, $1 \le j \le q+r$. Then for each j, N_j is a linear subspace of the space W_j. We set

$$J_0 = \{j \in J : 1 \le j \le q+r\}.$$

Define $N_{J_0} = \oplus_{j\in J_0} N_j$. If J_0 is empty, then let $N_{J_0} = \{0\}$.

THEOREM 3.1. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let μ be a (Q, b, α)-semi-stable centered Gaussian distribution, then*

$$\text{(3.1)} \qquad \textit{Spt } \mu \quad \textit{is a } B\textit{-invariant subspace of } N_{J_0}$$

and there uniquely exist centered Gaussian distributions μ_j, $j \in J_0$ such that Spt $\mu_j \subset N_j$ *for $j \in J_0$, and*

$$\text{(3.2)} \qquad \mu = *_{j \in J_0} \mu_j .$$

Where these distributions μ_j, $j \in J_0$, are (Q, b, α)-semi-stable .

THEOREM 3.2. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let $\sqrt{b^\alpha}$ be an eigenvalue of B and let $j \in J_0$ such that $\vartheta_j = \sqrt{b^\alpha}$. Then any centered Gaussian distribution μ with* Spt $\mu \subset N_j$ *is (Q, b, α)-semi-stable.*

For every $j (1 \leq j \leq q + 2r)$, we can choose $l_j \geq 1$, $z_{jl} \in V_j'$ and $n_{jl} \geq 1$ such that $(B' - \overline{\vartheta_j})^{n_{jl}} z_{jl} = 0$ and the system

$$\{z_{jln} = (B' - \overline{\vartheta_j})^{n-1} z_{jl} : 1 \leq l \leq l_j, 1 \leq n \leq n_{jl}\}$$

is basis of V_j'. Let us pick real z_{jl} for $1 \leq j \leq q$ so that $\{z_{jln} : 1 \leq l \leq l_j, 1 \leq n \leq n_{jl}\}$ is basis of W_j'. For $q+1 \leq j \leq q+r$, we have that $l_j = l_{j+r}$ and $n_{jl} = n_{j+r,l}$. We choose $z_{jl} \in V_j'$ for $q+1 \leq j \leq q+r$, $1 \leq l \leq l_j$ and define $z_{j+r,l}$ by $z_{j+r,l} = \overline{z_{jl}}$. Let u_{jln} and v_{jln} be the real and the imaginary part of z_{jl}, respectively for $q+1 \leq j \leq q+r$. Then we see that $\{u_{jln}, v_{jln} : 1 \leq l \leq l_j, 1 \leq n \leq n_{jl}\}$ is basis of W_j'. Let $J_1 = \{j \in J : 1 \leq j \leq q\}$ and let $J_2 = \{j \in J : q+1 \leq j \leq q+r\}$.

THEOREM 3.3. *Fix $b \in (0,1)$, $\alpha > 0$ and $Q \in M_+(R^d)$. Let $\mu \in I(R^d)$ with the Lévy representation $(\gamma, A, 0)$. Then μ is (Q, b, α)-semi-*

stable if and only if the following five conditions are satisfied:

(3.3)
$\phi_A(z) = 0 \quad \text{for} \quad z \in W'_j \quad j \notin J_0;$
(3.4)
$\langle Az, w\rangle = 0 \quad \text{for} \quad z \in W'_j, \quad w \in W'_k, \quad j \in J_0, k \in J_0, \quad j \neq k;$
(3.5)
$\phi_A(z_{jln}) = 0 \quad \text{for} \quad j \in J_1, \quad 1 \leq l \leq l_j, \quad 2 \leq n \leq n_{jl};$
(3.6)
$\phi_A(u_{jln}) = \phi_A(v_{jln}) \quad \text{for} \quad j \in J_2, \quad 1 \leq l \leq l_j, \quad 2 \leq n \leq n_{jl};$
(3.7)
$\langle Au_{jl1}, u_{jm1}\rangle = \langle Av_{jl1}, v_{jm1}\rangle$ and $\langle Au_{jl1}, v_{jm1}\rangle = -\langle Av_{jl1}, u_{jm1}\rangle$
for $j \in J_2, \quad 1 \leq l \leq l_j \quad 1 \leq m \leq l_j \quad (l = m$ *inclusive).*

THEOREM 3.4. *Fix* $b \in (0,1)$, $\alpha > 0$ *and* $Q \in M_+(R^d)$. *Fix* j *such that* $j \in J_0$ *and* $q + 1 \leq j \leq q + r$. *Let* μ *be a centered Gaussian distribution with covariance operator* A *with* Spt $\mu \subset N_j$. *If*

$$(3.8) \qquad \langle Au_{jl1}, u_{jm1}\rangle = \langle Av_{jl1}, v_{jm1}\rangle \quad \text{and} \quad \langle Au_{jl1}, v_{jm1}\rangle = -\langle Av_{jl1}, u_{jm1}\rangle \quad \text{for all } l \text{ and } m (l = m \ \text{inclusive}),$$

then μ *is* (Q, b, α)*- semi-stable.*

4. Proofs

For the proof of Theorem 3.1,we need the following lemma.

LEMMA 1. *Fix* $b \in (0,1)$, $\alpha > 0$ *and* $Q \in M_+(R^d)$. *Let* μ *be* (Q, b, α)*-semi-stable on* R^d*, and let* T *be a linear operator on* R^d *which commutes with* Q. *Then* $T\mu$ *is* (Q, b, α)*-semi-stable on* R^d.

Proof. From the assumption that $TQ = QT$, we see that

$$\begin{aligned}\widehat{T\mu}(z)^{b^\alpha} &= \widehat{\mu}(T'z)^{b^\alpha} = \widehat{\mu}(b^{Q'}T'z)e^{i\langle c(b), T'z\rangle} \\ &= \widehat{\mu}(T'b^{Q'}z)e^{i\langle c(b), T'z\rangle} = \widehat{T\mu}(b^{Q'}z)e^{i\langle Tc(b), z\rangle}. \qquad \square\end{aligned}$$

Proof of Theorem 3.1. Suppose that μ is a (Q, b, α)-semi-stable distribution with Lévy representation $(0, A, 0)$. Define $A_j = U_j A U_j'$ for $j \in J_0$. From the fact that

$$U_k A U_j' = 0 \quad \text{for} \quad j \neq k \tag{4. 1}$$

and

$$AU_j' = 0 \quad \text{for} \quad j \notin J_0,$$

we see that

$$A = \sum_{k,j=1}^{q+r} U_k A U_j' = \sum_j U_j A U_j' = \sum_{j \in J_0} A_j. \tag{4.2}$$

For $j \in J_0$, we see that $U_j A = A_j$ and $AU_j' = A_j$ by (4.1) and (4.2). Let μ_j be the centered Gaussian distribution with covariance A_j. Then (4.2) is (3.2). Noting that

$$\widehat{\mu_j}(z) = \exp\{-\frac{1}{2}\langle Az, z\rangle\} = \widehat{\mu}(U_j' z),$$

we see that $\mu_j = U_j \mu$. Hence μ_j is (Q, b, α)-semi-stable by Lemma 1, since $QU_j = U_j Q$ on R^d for $1 \leq j \leq q + r$. Using the same method in Theorem 4.2 of [9], we see that $A_j(R^d)$ is a B-invariant linear subspace of N_j.

Let us show the uniqueness of the decomposition (3.1) and (3.2). Let $\mu_j, j \in J_0$, be centered Gaussian distributions satisfying (3.1) and (3.2), and let B_j be the covariance operator of μ_j. Then $A = \sum_{j \in J_0} B_j$ and $B_j(R^d)$ =Spt $\mu \subset N_j$. Thus we have that $U_j B_j = B_j$ and $U_k B_j = 0$ for $j \neq k$. Hence $U_j A = B_j$ for $j \in J_0$. □

Proof of Theorem 3.2. Suppose that μ is a (Q, b, α)-semi-stable distribution with Lévy representation $(0, A, 0)$. Then we see that Sptμ $= A(R^d)$ by Lemma 4.2 in [9]. Using the fact that Spt $\mu \subset N_j$, we get $AT_k' = 0$ for $j \neq k$. Since Spt $\mu \subset N_j$, $(B - \vartheta_j)A = 0$, which gives that $(B - \vartheta_j)AT_j' = 0$. Hence we show (Q, b, α)-semi-stability of μ by Proposition 2.1. □

Proof of Theorem 3.3. Suppose that μ is a (Q, b, α)-semi-stable distribution with Lévy representation $(0, A, 0)$. Then the conditions (2.4) and (2.5) lead to (3.3) and (3.4). From (2.5), we get (3.5) and (3.6). We note that $\langle Az_{jl1}, \overline{z_{jm1}}\rangle = \langle Au_{jl1}, u_{jm1}\rangle - \langle Av_{jl1}, v_{jm1}\rangle + i\langle Au_{jl1}, v_{jm1}\rangle + \langle Av_{jl1}, u_{jm1}\rangle$, where $u_{jl1} + iv_{jm1} = z_{jl1} \in V_j'$ and $u_{jl1} - iv_{jm1} = \overline{z_{jl1}} \in V_{j+r}'$. Using this and (2.5), we also get (3.7).

Conversely, suppose that A satisfies (3.3),(3.4),(3.5), (3.6) and (3.7). In case $j \notin J$, we see that $Az = 0$ for $z \in V_j'$ by (3.3). Hence we get (2.4). In case $j \in J$, we can prove (2.5) by the same proof as that of Theorem 4.1 of [9]. □

Proof of Theorem 3.4. Let $z \in W_k'$, $k \neq j$. Then we have $Az \in N_j \subset W_j$. So $Az = 0$, since W_j and W_k' are orthogonal. Hence (3.3), (3.4) and (3.5) hold. Let $j \in J_2$ and $x \in N_j$. Then $(B - \vartheta_j)(B - \overline{\vartheta_j})x = 0$. Hence there are x_1 and x_2 in C^d such that $x = x_1 + x_2$, $(B - \vartheta_j)x_1 = 0$ and $(B - \overline{\vartheta_j})x_2 = 0$. For $n \geq 2$, we have $\langle x_1, z_{jln}\rangle = 0$ and $\langle x_2, z_{jln}\rangle = 0$. Hence $\langle x, z_{jln}\rangle = 0$. Thus (3.6) is proved. By the above hypothesis, (3.7) also holds. Thus μ is (Q, b, α)-semi-stable. □

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