1122 *KSME International Journal, Vol. 14, No. 10, pp. 1122~1130, 2000*

A New Penalty Parameter Update Rule in the Augmented Lagrange Multiplier Method for Dynamic Response Optimization

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Based on the value of the Lagrange multiplier and the degree of constraint activeness, a new update rule is proposed for penalty parameters of the ALM method. The theoretical exposition of this suggested update rule is presented by using the algorithmic interpretation and the geometric interpretation of the augmented Lagrangian. This interpretation shows that the penalty parameters can effect the performance of the ALM method. Also, it offers a lower limit on the penalty parameters that makes the augmented Lagrangian to be bounded. This lower limit forms the backbone of the proposed update rule. To investigate the numerical performance of the update rule, it is embedded in our ALM based dynamic response optimizer, and the optimizer is applied to solve six typical dynamic response optimization problems. Our optimization results are compared with those obtained by employing three conventional update rules used in the literature, which shows that the suggested update rule is more efficient and more stable than the conventional ones.

Key Words : Penalty Parameter, Augmented Lagrange Multiplier Method, Dynamic Response Optimization

1. Introduction

The optimal design problems of many mechanical systems and seismic-resistant structures are often mathematically modeled as a dynamic response optimization problem. It is the parameter dependency and implicit nature of parametric constraint functions that make the dynamic response optimization problem difficult and expensive to solve. Many works (Haug and Arora, 1979; Hsieh and Arora, 1984; Hsieh and Arora, 1985; Paeng and Arora, 1989; Chahanda and Arora, 1994; Kim and Choi, 1998) have focused on efficiently dealing with parametric constraint functions. Among these works, the approaches (Paeng and Arora, 1989; Chahanda and Arora, 1994; Kim and Choi, 1998) that employ the Augmented Lagrange Multiplier (ALM) method showed more efficient results for dynamic response optimization of large scale systems than the other approaches (Haug and Arora, 1979; Hsieh and Arora, 1984; Hsieh and Arora, 1985) based on the primal optimization algorithms.

In the conventional ALM method, however, penalty parameters are generally increased in the same manner as for the exterior penalty function method. Although the ALM method is known to be relatively insensitive to the values of a penalty parameter, its efficiency may depend on the penalty parameter update rule because the penalty parameter is embedded in the Lagrange multiplier update rule.

This study is devoted to the development of a penalty parameter update rule which can improve the efficiency of the ALM method. The update rule is theoretically based on the duality theory

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and the geometric interpretation of the augmented Lagrangian. It is also conceptually founded on the virtue of adaptation to Lagrange multiplier values and constraint activeness which vary during the optimization process.

In Sec. 2, the ALM method for dynamic response optimization is presented and the role of penalty parameters in the ALM method is investigated. In sec. 3, a new penalty parameter update rule is proposed. In sec. 4, six typical dynamic response optimization problems are solved by using an ALM optimizer in which the proposed penalty parameter update rule is embedded. Then, optimization results are compared with those obtained by using the three conventional update rules used in the open literature. Finally, concluding remarks are mentioned.

2. The ALM Method for Dynamic Response Optimization

2.1 Augmented Lagrangian

To present the general idea of a new penalty parameter update rule in the ALM method for dynamic response optimization, a simplified model of the dynamic response optimization problem is considered. It is understood that the update rule presented here is applicable to more complex models.

The dynamic response optimization problem is defined as follows:

$$\begin{aligned}&\text{minimize } \Psi_0(b, z)\\&\text{subject to } \Psi_i(b, z, \alpha)\\&\quad \le 0 \text{ for } 0 \le \alpha \le \Lambda,\ i=1, \ldots, m\end{aligned} \tag{1}$$

where $b \in R^n$ is a vector of design variables, $z \in R^k$ is a vector of generalized velocities and displacements, and α is an environmental parameter such as time or frequency. Ψ_0 is the cost function and Ψ_i is the i^{th} parametric constraint function that must hold over the entire parameter interval $[0, \Lambda]$.

In general optimization problems, the Lagrangian might not be convex near a solution, and hence the duality method can not be applied (Luenberg, 1984; Gill et al., 1981). Thus, the penalty term is added to the Lagrangian in order to make the functional (augmented Lagrangian) convex. This augmented Lagrangian is sequentially minimized by using well-developed unconstrained optimization algorithms, which is collectively known as the ALM method.

The augmented Lagrangian can be defined in various ways (Arora et al, 1991). In this study, we exploit the augmented Lagrangian suggested by Rockafeller (1973) and extend it to treat parametric constraints as

$$\begin{aligned}A(b, z, \mu, r) = \Psi_0(b, z) + \int_0^\Lambda \sum_{i=1}^m [\mu_i(\alpha)\\ \Omega_i(b, z, \alpha) + \frac{1}{2} r_i \Omega_i(b, z, \alpha)^2] d\alpha\end{aligned} \tag{2}$$

where $\Omega_i(b, z, \alpha) = \max[\Psi_i(b, z, \alpha), -\mu_i(\alpha)/r_i]$; $\mu_i(\alpha) \ge 0$ and $r_i > 0$ are the Lagrange multiplier function and the penalty parameter for the i^{th} parametric constraint, respectively. We also extend Rockafeller's Lagrange multiplier update rule to handle parametric constraints as

$$\begin{aligned}&\mu_i^{k+1}(\alpha) = \mu_i^k(\alpha) + r_i^k \cdot \Omega_i(b, z, \alpha) \text{ for}\\&0 \le \alpha \le \Lambda,\ i=1, \ldots, m\end{aligned} \tag{3}$$

2.2 Interpretation of an augmented Lagrangian

2.2.1 Algorithmic interpretation

From the viewpoint of the duality theory, an augmented Lagrangian can be interpreted as the Lagrangian for the following problem:

$$\begin{aligned}&\text{minimize } \Psi_0(b, z) + \int_0^\Lambda \sum_{i=1}^m \left[\frac{1}{2} r_i \Omega_i(b, z, \alpha)^2\right] d\alpha\\&\text{subject to } \Psi_i(b, z, \alpha) \le 0 \text{ for } 0 \le \alpha \le \Lambda,\ i=1, \ldots, m\end{aligned} \tag{4}$$

This problem is equivalent to the original problem of Eq. (1), since the addition of a penalty term to a cost function does not change an optimal value. However, whereas the original Lagrangian may not be convex near a solution, a penalty term tends to make the Lagrangian for Eq. (4) convex. For sufficiently large penalty parameter values, this Lagrangian will indeed be locally convex. This viewpoint leads to an idea of limiting penalty parameter values from below when they are updated.

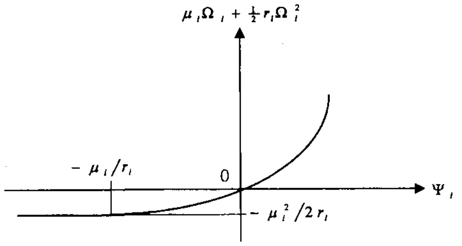

Fig. 1 Geometric interpretation of μ_i/r_i for the i^{th} parametric constraint

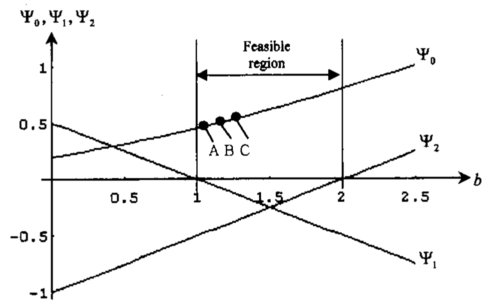

Fig. 2 Cost and constraint functions of optimization problem of Eq. (5)

2.2.2 Geometric interpretation

The augmented Lagrangian of Eq. (2) suggests an interesting interpretation for the role of (μ_i/r_i)'s. The initial values of μ_i's are usually set at zero, which makes Eq. (2) the same as the pseudo-cost function of the exterior penalty function method. At this stage, only the constraints with positive values are penalized. For the next stage, some (μ_i/r_i)'s may become positive when the corresponding constraints are positive, according to the update rule of Eq. (3). The positive μ_i/r_i pushes the threshold level for penalizing the corresponding constraint into the feasible region as shown in Fig. 1. This push penalizes not only the constraints with positive values but also the constraints with negative values that are greater than the corresponding $-\mu_i/r_i$. In subsequent stages, further increase or decrease of the value of μ_i/r_i depends on the value of the corresponding constraint, which can be easily understood from the Lagrange multiplier update rule of Eq. (3).

To better understand the interpretation, consider the following optimization problem with one variable:

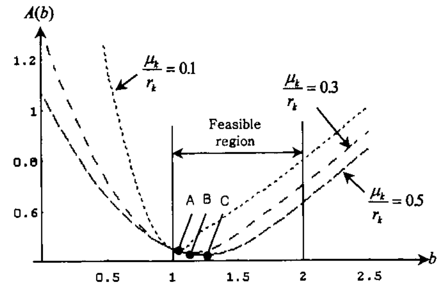

Fig. 3 Augmented Lagrangians for various values of the threshold level

$$\text{minimize} \quad \Psi_0(b) = \frac{(b+2)^2}{20} \tag{5a}$$

$$\text{subject to} \quad \Psi_1(b) = (1-b)/2 \le 0 \tag{5b}$$

$$\Psi_2(b) = (b-2)/2 \le 0 \tag{5c}$$

The cost and constraint functions are plotted in Fig. 2 and the constrained minimum is clearly $\Psi_0^* = 0.45$ at $b=1$. Now, the augmented Lagrangian function becomes

$$A(b, r, \mu) = \frac{(b+2)^2}{20} + \mu_1 \cdot \max\left\{\frac{1-b}{2}, -\frac{\mu_1}{r_1}\right\} + \frac{1}{2} \cdot r_1\left[\max\left\{\frac{1-b}{2}, -\frac{\mu_1}{r_1}\right\}\right] + \mu_2 \cdot \max\left\{\frac{b-2}{2}, -\frac{\mu_2}{r_2}\right\} + \frac{1}{2} \cdot r_2\left[\frac{b-2}{2}, -\frac{\mu_2}{r_2}\right]^2 \tag{6}$$

Figure 3 graphically shows the augmented Lagrangian for various values of the threshold levels. The result implies that the augmented Lagrangian having a larger threshold level (design point C) leads to a more excessively feasible design than those having smaller threshold levels (design points A and B). Thus, it can be shown that the threshold level requires a lower limit in order to avoid an excessively feasible design.

2.3 Computational procedure of the ALM method

Although many different versions of the ALM methods are now available, the following algorithm is employed for solving the dynamic response optimization problem in this study:

Step 1 Select an initial design variable vector b^0, an initial Lagrange multiplier vector μ^0, and an initial penalty parameter vector r^0. Set $q=0$.

Step 2 Starting from b^q, minimize $A(b, z, \mu^q, r^q)$ of Eq. (2) subject to $b^L \leq b \leq b^u$, where b^L and b^U are the vectors of the lower and upper limit values on design variables, respectively (Kim and Choi, 1995). These side constraints are considered separately for the design variables so as not to have physically unrealistic values . Let the solution be b^{q+1}.

Step 3 At the optimum b^{q+1}, if the peak of every point-wise state variable constraint is lower than a specified tolerance ε and the relative reduction of the cost value is less than a specified tolerance ε_c, then stop. Otherwise, go to Step 4.

Step 4 Update the Lagrange multipliers by Eq. (3), then update the penalty parameters based on the Lagrange multiplier values and the degrees of satisfaction of the constraint functions. Go to Step 2 with $q=q+1$.

In Step 4, the penalty parameters should be updated to suitable values. If they are too small, there is not only the danger of unboundness of the augmented Lagrangian but also the possibility of ill-conditioning in the unconstrained subproblem. In other words, they should be bounded from below for the subproblem to have a local minimum. If they are too large, the phenomenon of an ill-conditioned subproblem occurs as in the exterior penalty function method. Thus, we elaborate on our approach to the penalty parameter update rule in the next section.

3. A New Update Rule for Penalty Parameters

3.1 Basic concept

The interpretation of Sec. 2 implies that the penalty parameters require the lower limit in order to make the augmented Lagrangian convex and to avoid an excessively feasible design.

In order to fundamentally determine the convexity of a function, the second derivatives of the function are required. However, it is not practically recommended due to the computational efforts involved in obtaining them. Hence, in order to enforce only the role of a penalty term in the penalty-like term during line search, we suggest that the penalty term should satisfy the following conditions:

$$\frac{1}{2} r_i \Psi_i^2 \geq \mu_i \Psi_i, \; i=1, 2, \ldots, m \tag{7}$$

As the iterative design goes to the constrained optimum, the value of Ψ_i^2 is much less than that of Ψ_i since $\Psi_i \ll 1$. Therefore, the augmented Lagrangian may not be bounded near the optimum. Thus, Eq. (7) is especially useful in making the augmented Lagrangian convex in the vicinity of an optimum. The condition (7) offers the lower bound on the penalty parameters as

$$r_i \geq \frac{2\mu_i}{\Psi_i}, \; i=1, 2, \ldots, m \tag{8}$$

If the value of Ψ_i is negative or nearly zero, however, the condition (8) includes a possibility that the penalty parameters become negative or infinity. Thus, we modify the condition (8) as follows:

$$r_i \geq \frac{2\mu_i}{\max\{\Psi_i, \delta_1\}}, \; i=1, 2, \ldots, m \tag{9}$$

where δ_1 is an appropriate positive value. Also, the value of $\max\{\Psi_i, \delta_1\}$should be less than $2\mu_i$ since the penalty parameter should be intrinsically greater than 1 in the augmented Lagrangian of Eq. (2).

Also, from the viewpoint of shifting the threshold level, the shifting parameter μ_i/r_i is recommended to be less than a suitable positive value to avoid an excessively feasible design. This concept offers us a rough lower bound on the penalty parameter as

$$r_i \geq \frac{\mu_i}{\delta_2}, \; i=1, 2, \ldots, m \tag{10}$$

The value of δ_2 should be greater than zero and less than μ_i, for the same reasons as for δ_1 in Eq. (9). Consequently, the conditions (9) and (10) can be combined as:

$$r_i \geq \frac{\mu_i}{\delta}, \; i=1, 2, \ldots, m \tag{11}$$

where the value of δ is an appropriate value in $(0, \mu_i)$. In general, a smaller value of δ is preferred as long as a larger penalty parameter does not make the augmented Lagrangian stiff in the design variable space.

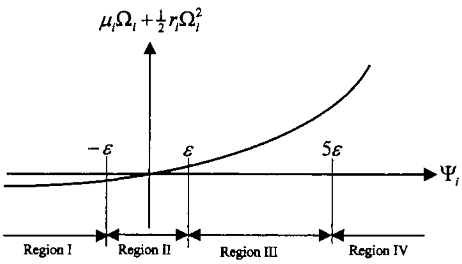

Fig. 4 The constraint domain divided for implementing the activeness of constraints in updating penalty parameter

Table 1 The values of ζ for implementing the activeness of constraints

Regions	Conditions	The values of ζ
I	$\mu_i^{k+1} \neq 0$	$\min\{\beta_1 \cdot r_i^k, \mu_i^{k+1}/\varepsilon\}$
	$\mu_i^{k+1}=0$	$\beta_1 \cdot r_i^k$
II	$\Psi_i \geq 0$	μ_i^{k+1}/ε
	$\Psi_i < 0$	$\frac{1}{10}\mu_i^{k+1}/\varepsilon$
III	-	$\max\{\beta_1 \cdot r_i^k, \mu_i^{k+1}/\varepsilon\}$
IV	-	$\min\{\beta_2 \cdot r_i^k, \mu_i^{k+1}/\varepsilon\}$

3.2 Numerical implementation

In this study, we empirically divide constraint domains into four regions as shown in Fig. 4 and individually update the penalty parameter for each of the four regions. We define Region I as a feasible domain, Region II as an active domain, Region III as a slightly violated domain, and Region IV as a severely violated domain. In Fig. 4, the positive value of ε is specified by a user defined convergence tolerance for parametric constraints.

As the optimization progresses, violated constraints may be fully satisfied. In this case, their corresponding Lagrange multipliers may be nearly zero. Thus, the value of δ can not be determined by using only the Lagrange multiplier. Therefore, we propose two separate penalty parameter update rules that depends on whether the Lagrange multipliers are zero or not. Also, we suggest that the value of δ is specified by using a user defined convergence tolerance ε, which is based on the following two reasons:

- as the violated constraints are destined to be less than ε at the optimum, the value of ε will be the final value of the violated constraints.
- as mentioned in Sec. 2, a smaller value is preferred for δ as long as the larger penalty-like term does not make the augmented Lagrangian stiff.

In order to avoid the numerically ill condition due to a larger penalty parameter, we use a problem scaling scheme to weigh the magnitude of a cost function and that of the penalty-like term. The proposed penalty parameter update rule is

$$r_i^{k+1}=\max\{2r_i^k, \zeta\} \tag{12}$$

The value of are separately recommended for each of the four Regions, which are defined by using the lower bound of Eq. (11). They are summarized in Table 1. Also, the maximum scheme and $2r_i^k$ in Eq. (12) represent safeguards for maintaining the basic role of the penalty parameters.

In Table 1, β_1 and β_2 are recommended to have the values of 5 and 10, which are introduced to compensate ζ when the Lagrange multiplier is zero and to accelerate the feasibility for violated constraints. The values of ζ are proposed as follows

- in Region I, the constraints may have zero Lagrange multipliers in the feasible region. Thus, the update rules are defined separately by checking whether the Lagrange multipliers are zero or not. If the Lagrange multiplier is zero, the update rule follows the conventional concept. Otherwise, the constraints have the possibility to be more active. Hence, we employ the minimum between the value evaluated by the conventional update rule and that by the suggested update rule.
- in Region II, the constraints are active. Thus, their penalty parameters are directly updated by using condition (11). In this case, we want the constraints in the interval $(-\varepsilon, 0)$ to be more active. Thus, the corresponding penalty parameters are selected to be less than that of the constraints in the interval $(0, \varepsilon)$.

● in Region III, the constraints are slightly violated. In this case, we want to accelerate the feasibility of the constraints. Hence, we employ the maximum value between the value evaluated by the conventional update rule and that by the suggested update rule.

● in Region IV, the constraints are severely violated. Hence, the corresponding Lagrange multipliers may be inaccurate and quite large because the constraint is remote from an optimum. In this case, we use the minimum scheme to offer relatively larger shifting thresholds.

4. Numerical Experiments and Discussions

The performance of the proposed penalty parameter update rule is numerically investigated by solving six typical dynamic response optimization problems and comparing the results with those of the conventional update rule in which the penalty parameters are imreased by a factor β ($r_i^{k+1}=\beta \cdot r_i^k$, $i=1, ..., m$). The specific β values selected in this study are 3, 5 and 10.

These four penalty parameter update rules are implemented in our dynamic response optimizer IDOL 3.0 (Kim, 1997) based on the ALM method in which the BFGS method is employed for finding descending directions and a sequential polynomial approximation method (Kim and Choi, 1995) imployed for line search. Also, the max-value cost function over time interval is directly handled in these optimizations (Kim and Choi, 2001).

The same numerical procedures are employed except for the penalty parameter update rule for direct comparison of the efficiency of the proposed update rule with those of the conventional ones. The Runge-Kutta-Verner fifth-and sixth-order method is used for dynamic analysis and the direct differenciation method for design sensitivity analysis. Simpson's rule is used for the integration of Eq. (2). The convergence tolerance (ε, ε_C) are specified as 1.0×10^{-4} and 1.0×10^{-1} in all sample problems.

The six typical dynamic response optimization problems solved in this study are four design cases of one degree of freedom nonlinear impact absorber, one design case of two degrees-of-freedom dynamic absorber, and one design case of five degrees-of-freedom vehicle suspension system. One may refer to Haug and Arora (1979) for detailed information.

4.1 Single degree-of-freedom nonlinear impact absorber

A single degree-of-freedom nonlinear impact

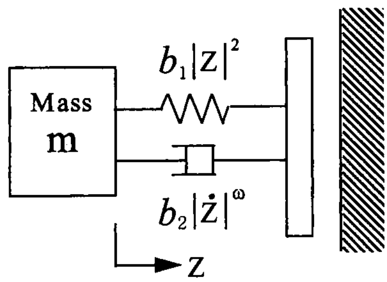

Fig. 5 Nonlinear impace absorber

Table 2 Comparisons of optimization results for the single degree-of-freedom impact absorber

(a) $\omega=1$

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	0.5283	0.5283	0.5433	0.5282
NG	11	48	28	32
NF	37	13	84	108

(b) $\omega=2$

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	0.5972	0.5971	0.5972	0.5969
NG	16	24	25	21
NF	47	66	76	57

(c) $\omega=3$

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	0.6833	0.6832	0.6832	0.6829
NG	14	28	24	22
NF	46	80	69	62

(d) $\omega=4$

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	0.7540	0.7540	0.7539	0.7540
NG	14	27	22	21
NF	40	77	62	61

absorber of Fig. 5 has a fixed mass and two design variables (b_1, b_2) that represent spring and damper coefficients, respectively. The system impacts a fixed barrier at time $t=0$ with a given initial velocity. The objective is to find b_1 and b_2 that minimize the maximum acceleration of the mass, subject to a constraint on extreme displacement. This example problem is solved for $\omega=1$, 2, 3 and 4. The initial design values of the design variables are selected as [0.50, 0.50]. The optimization results obtained by using the four update rules are listed in Table 2, where NG denotes the number of gradient evaluations, NF the number of function evaluations and COST the optimum value of a cost function.

As can be seen in Table 2, the proposed update rule is more efficient than the conventional ones in all four design cases. The proposed update rule can reduce the gradient evaluations by 36~60% percents and the function evaluations by 35~55%.

4.2 Linear two degree-of-reedom vibration isolator

A two degree-of-freedom linear dynamic absorber is shown in Fig. 6. The objective is to find the damping and spring constants{k, c} that minimize the peak transient dynamic displacement of the main mass for a given excitation frequency, subject to constraints on transient and steady state responses and explicit bounds on the design variables. The optimization results

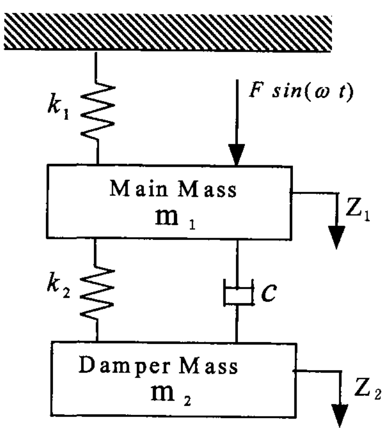

Fig. 6 Vibration isolator

obtained by using the four update rules are listed in Table 3. The initial design values of the design variable are selected as [1.60, 0.02] and the corresponding initial cost value is 3.182.

Table 3 shows that the proposed update rule is more efficient than the conventional ones. In this example problem, we can see that the subsequent iterations of the optimization process by using the proposed update rule are always in feasible or active regions through all the ALM iterations. However, when the conventional update rules are used, their subsequent iterations are sometimes in the violated regions.

4.3 Five degree-of-freedom vehicle suspension system

Figure 7 shows a five degree-of-freedom vehicle suspension system, which is to be designed to minimize the extreme acceleration of the driver's seat for a given vehicle speed and road surface profile shown in Fig. 8. Spring constants k_1, k_2 and k_3 and damping coefficients c_1, c_2 and c_3 of the system are chosen as the design variables. The motion of the vehicle is constrained so that the relative displacements between the chassis and driver's seat, the chassis and the front and rear axles, and the road surface and front and rear axles are within given limits. The design variables are also constrained. The optimization results

Table 3 Comparisons of optimization results for the two degree-of-freedom impact absorber

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	2.3559	2.3649	2.3613	2.3553
NG	13	16	14	20
NF	41	55	54	67

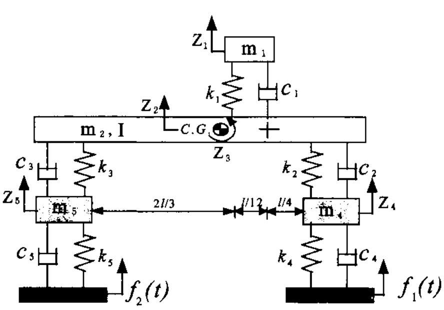

Fig. 7 Five degree of freedom vehicle model

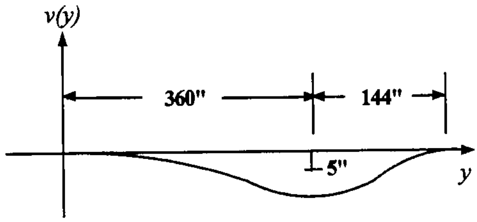

Fig. 8 Road surface profile

'able 4 Comparisons of optimization results for the five dof vehicle system

	Proposed	$\beta=3$	$\beta=5$	$\beta=10$
COST	254.90	255.51	256.99	255.53
NG	17	29	27	30
NF	51	90	88	86

›btained by using the four update rules are listed n Table 4. The starting values of the design 'ariable are [100.0, 300.0, 300.0, 10.0, 25.0, 25.0] ınd the corresponding initial cost value is 331. 8.

The proposed update rule shows good effi:iency similar to the above two examples. Even, n this medium-size problem, the proposed up-late rule can reduce the gradient evaluations by 37～43% and the function evaluations by 40 ～43%.

5. Concluding Remarks

In this study, a new penalty parameter update rule which is based on the value of the Lagrange multiplier and the degree of constraint activeness, is presented for the ALM method. The theoretical background of the suggested update rule is presented by using the duality theory and the geometric interpretation of the augmented Lagrangian. The scheme offers a lower bound on the penalty parameter to make the augmented Lagrangian convex and to avoid an excessively feasible design, and represents fundamental concepts of the proposed penalty parameter update rule.

Also, in order to investigate the numerical performance, this update rule is embedded in the ALM based dynamic response optimizer. This optimizer solves six typical dynamic response optimization problems and the optimization results are compared with those of other optimizers employing three conventional update rules. These comparisons show that the proposed update rule is more efficient than the conventional ones in all six tested problems. Even in a medium-sized problem, the proposed update rule can reduce the gradient evaluations by 37～43% and the function evaluations by 40～43%, compared with the conventional ones.

Acknowledgement

This research was supported by center of Innovative Design Optimization Technology (iDOT), Korea Science and Engineering Foundation and by the BK21 projects, division for research and education in mechanical engineering of the Korean ministry of education.

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