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THE EXISTENCE THEOREM OF ORTHOGONAL MATRICES WITH p NONZERO ENTRIES

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ABSTRACT. It was shown that if Q is a fully indecomposable $n \times n$ orthogonal matrix then Q has at least $4n-4$ nonzero entries in 1993. In this paper, we show that for each integer p with $4n-4 \leq p \leq n^2$, there exist a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries. Furthermore, we obtain a method of construction of a fully indecomposable $n \times n$ orthogonal matrix which has exactly $4n-4$ nonzero entries. This is a part of the study in sparse matrices.

1. Introduction

By a *pattern* we simply mean the arrangement of zero and nonzero (denoted *) entries in a matrix. An $n \times n$ pattern $\mathcal{A}$ is called *orthogonal* if there is an orthogonal matrix A whose pattern is $\mathcal{A}$. An $n \times n$ matrix A is *fully indecomposable*, if the rows and columns of A can not be permuted to obtain a matrix of the form

$$\begin{bmatrix} A_{11} & O \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} and A_{22} are square and nonempty. Let $\#(A)$ denote the number of nonzero entries in the matrix (or pattern) A. In [BBS], it was shown that if Q is a fully indecomposable $n \times n$ orthogonal matrix then $\#(Q) \geq 4n-4$. Thus it is clear that if Q is a fully indecomposable $n \times n$ orthogonal matrix then

$$4n-4 \leq \#(Q) \leq n^2.$$

Equivalently, if Q is a fully indecomposable $n \times n$ orthogonal matrix then the number of zeros in Q is between 0 and $(n-2)^2$. In [CJLP], it was studied the possible numbers of zeros in an orthogonal matrix.

In this paper, we show that for each integer p with $4n-4 \leq p \leq n^2$, there exist a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries. We introduce a constructive approch for such matrices.

2. Orthogonal pattern by symbolic Gram-Schmidt procedure

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We say that an $n \times n$ matrix A (or pattern $\mathcal{A}$) satisfies the *Hall property* if every k columns, $1 \leq k \leq n$, collectively have nonzero entries in at least k rows. It is clear that there are full rank matrices with pattern $\mathcal{A}$ if and only if $\mathcal{A}$ satisfies the Hall property. Note that every lower triangular pattern such that its each main diagonal entry is nonzero satisfies the Hall property. The Gram-Schmidt orthonormalization process can be used to construct the matrix Q of the QR factorization of a matrix A. In [HJOD], it was given how to adapt the *symbolic Gram-Schmidt procedure* to determine $\mathcal{Q}$ for a pattern $\mathcal{A}$ under the assumption that $\mathcal{A}$ has the Hall property.

For a $k \times k$ pattern $\mathcal{L} = [l_{ij}]$ defined by

$$l_{ij} = \begin{cases} 0 & \text{if } \ i < j, \\ * & \text{if } \ i = j \ \text{ or } \ j = i-1, \\ 0 \text{ or } * & \text{otherwise}, \end{cases}$$

we say that $\mathcal{L}$ is in *echelon form* whenever, for each i and j with $i \geq j+2$, $l_{ij} = 0$ then $l_{i+1\,j} = \cdots = l_{kj} = 0$ and $l_{i1} = \cdots = l_{i\,j-1} = 0$ for $j \geq 2$.

Clearly, each pattern $\mathcal{L}$ in echelon form is lower triangular such that its each main diagonal entry is nonzero. For example, if $m = 4$ then each pattern in echelon form is following:

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & * \end{bmatrix}, \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix}, \text{or} \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{bmatrix}.$$

Since each pattern $\mathcal{L}$ in echelon form satisfies Hall property, the next lemma follows from [HJOD].

LEMMA 2.1. *Let $\mathcal{L}$ be an $n \times n$ pattern in echelon form. If $\mathcal{Q}$ is a pattern obtained from $\mathcal{L}$ by symbolic Gram-Schmidt procedure, then $\mathcal{Q}$ is the same as $\mathcal{L}+\mathcal{U}$, and $\mathcal{Q}$ is a fully indecomposable $n \times n$ orthogonal pattern, where $\mathcal{U} = [u_{ij}]$ is the $n \times n$ pattern with*

$$u_{ij} = \begin{cases} * & \textit{if } \ i < j, \\ 0 & \textit{otherwise}, \end{cases}$$

and $\mathcal{L}+\mathcal{U}$ is a pattern which has a zero in a given position only when both patterns are zero in that position.

Note that if we take $\mathcal{L}$ as the $n \times n$ full lower triangular pattern and the $n \times n$ pattern

$$\begin{bmatrix} * & & & & \\ * & * & & O & \\ & * & \ddots & & \\ & O & \ddots & \ddots & \\ & & & * & * \end{bmatrix}$$

respectively, then by Lemma 2.1, we can get the $n \times n$ orthogonal full pattern $\mathcal{Q}$ with $\#(\mathcal{Q}) = n^2$ and the $n \times n$ orthogonal upper Hessenberg pattern $\mathcal{Q}$ with $\#(\mathcal{Q}) = (n^2+3n-2)/2$, respectively, where a matrix is full means all entries in the matrix are nonzero.

THEOREM 2.2. *For each integer p with*

$$\frac{n^2+3n-2}{2} \leq p \leq n^2, \tag{1}$$

there is a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries.

Proof. Let $\mathcal{L}$ be an $n \times n$ pattern in echelon form, and let $\mathcal{Q}$ be a pattern obtained from $\mathcal{L}$ by symbolic Gram-Schmidt procedure. Then by Lemma 2.1, $\mathcal{Q}$ is an $n \times n$ fully indecomposable orthogonal pattern. Furthermore,

$$\#(\mathcal{Q}) = \#(\mathcal{L}) + \#(\mathcal{U}) = \#(\mathcal{L}) + \frac{(n-1)n}{2}.$$

It is easy to show that for each integer q with $2n-1 \leq q \leq \frac{n(n+1)}{2}$, there is a pattern $\mathcal{L}$ in echelon form with exactly q nonzero entries. Thus the theorem follows. ■

Now, we only need to show that for each integer p with

$$4n-4 \leq p \leq \frac{n^2+3n-2}{2} - 1 \tag{2}$$

there is a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries. In order to do this, we shall use the method of weaving and woven matrix which can be found in [Cr].

3. Orthogonal pattern by weaving

In [Cr], R. Craigen introduced a matrix method called *weaving* which is a method for building new matrices from the given one. To every $(0,1)$-matrix $A = [a_{ij}]$ having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$, we adopt the following notations.

$s = s(i,j) :=$ the number of nonzero positions in ith row of A up to jth column,
$t = t(i,j) :=$ the number of nonzero positions in jth column of A up to ith row.

Now let R_i $(i = 1, 2, \ldots, m)$ have r_i columns $\mathbf{u}_s$, and C_j $(j = 1, 2, \ldots, n)$ have c_j rows $\mathbf{v}_t^T$ using the indices s and t introduced above. We define the "*woven product*" $M = (R_1 \cdots R_m) \circledast (C_1 \cdots C_n) = [M_{ij}]$ block entrywise by

$$M_{ij} = \begin{cases} \mathbf{u}_s \mathbf{v}_t^T & \text{if} \quad a_{ij} = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The matrix A is called the *lattice* of the weaving, and the matrix M obtained by weaving is called a *woven* matrix. Note that the resulting woven matrix depends on the lattice and matrices R_i's, C_j's.

Throughout we let $M(A)$ denote a woven matrix obtained by weaving from a given lattice A.

LEMMA 3.1. *[C]* *Let A be an $m \times n$ lattice having row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_n$ whose bipartite graph is connected. If R_i $(i = 1, 2, \ldots, m)$ is a fully indecomposable $r_i \times r_i$ orthogonal matrix and C_j $(j = 1, 2, \ldots, n)$ is a fully indecomposable $c_j \times c_j$ orthogonal matrix, then the woven matrix $M(A)$ is fully indecomposable orthogonal matrix of order $\#(A)$.*

Now, we are ready to construct orthogonal matrices with p nonzero entries in (2) by weaving.

We are going to prove for odd n and for even n, respectively. Let us consider the odd $n = 2m - 1$ case first.

For each $k = 2, \ldots, m$, we define the $m \times m$ (0,1)-matrix $A_k = [a_{ij}]$ with row sums $r_1, r_2, \ldots, r_m$ and column sums $c_1, c_2, \ldots, c_m$ by

$$a_{ij} = \begin{cases} 1 & \text{if } i = j;\ i = 1 \text{ and } j = 2, \ldots, k; \\ & \quad j = i + 1 \text{ for } i = k, \ldots, m - 1, \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$A_k = \begin{bmatrix} 1 & \overbrace{1 & \cdots & \cdots & 1}^{k \text{ times}} & 0 & \cdots & 0 \\ & 1 & 0 & & & & & \\ & & \ddots & \ddots & & & O & \\ & & & 1 & 0 & & & \\ & & & & 1 & 1 & & \\ & & O & & & 1 & \ddots & \\ & & & & & & \ddots & 1 \\ & & & & & & & 1 \end{bmatrix}_{m \times m}. \qquad (3)$$

Thus $\#(A_k) = 2m - 1$ and $r_1 = k,\ r_2 = \cdots = r_{k-1} = 1,\ r_k = \cdots = r_{m-1} = 2,\ r_m = 1, c_1 = 1,\ c_2 = \cdots = c_m = 2$. Take

$$R_1 = \begin{bmatrix} * & \cdots & * \\ \vdots & \vdots & \vdots \\ * & \cdots & * \end{bmatrix}_{k \times k},$$

$$R_2 = \cdots = R_{k-1} = R_m = C_m = [\,*\,], \qquad (4)$$

$$R_k = \cdots = R_{m-1} = C_2 = \cdots = C_m = \begin{bmatrix} * & * \\ * & * \end{bmatrix}. \qquad (5)$$

Then R_i's and C_j's are orthogonal patterns from Lemma 2.1. And since A_k has a connected bipartite graph, the resulting woven matrix $M(A_k)$ is fully indecomposable $(2m-1) \times (2m-1)$

orthogonal pattern from Lemma 3.1, and has the following form:

$$
\left[\begin{array}{c|cc|c|cc|cc|cc|cc|c|cc|cc}
* & * & * & & * & * & * & * & & & & & & & & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & \\
* & * & * & & * & * & * & * & & & & & & & & \\ \hline
 & * & * & & & & & & & & & & & & & \\ \hline
 & & & \ddots & & & & & & & & & & & & \\ \hline
 & & & & * & * & & & & & & & & & & \\ \hline
 & & & & & & * & * & * & * & & & & & & \\
 & & & & & & * & * & * & * & & & & & & \\ \hline
 & & & & & & & & * & * & * & * & & & & \\
 & & & & & & & & * & * & * & * & & & & \\ \hline
 & & & & & & & & & & & & \ddots & & & \\ \hline
 & & & & & & & & & & & & & * & * & * & * \\
 & & & & & & & & & & & & & * & * & * & * \\ \hline
 & & & & & & & & & & & & & & & * & *
\end{array}\right].
$$

Thus

$$
\begin{aligned}
\#(M(A_k)) &= k(2k-1) + 2(k-2) + 4(n-k-(k-2)-1) + 2 \\
&= 4n - 4 + (2k^2 - 7k + 6).
\end{aligned}
$$

Remark 3.2. *If $k = 2$ in (3) then $\#(M(A_k)) = 4n - 4$ and $M(A_2)$ is an unique orthogonal pattern for an odd n as noted in [BBS].*

Note that if we fix the matrices R_i's ($i \neq 1$) and C_j's in (4) and (5) then $\#(M(A_k))$ depends on $k \times k$ orthogonal pattern R_1. Now, fix R_i's ($i \neq 1$) and C_j's in (4) and (5), and suppose R_1 has a zero entry. Let x_i denote the number of zero entries in the ith column of R_1. Then it is clear that

$$
\#(M(A_k)) = 4n - 4 + (2k^2 - 7k + 6) - (x_1 + 2x_2 + 2x_3 + \cdots + 2x_k). \tag{6}
$$

Lemma 3.3. *For an integer $k \geq 3$, let x_i be an integer such that*

$$
0 \leq x_i \leq k - i - 1, \quad (i = 1, 2, \ldots, k-2).
$$

Then the linear system

$$
x_1 + 2x_2 + \cdots + 2x_{k-2} = t \quad \textit{with} \quad x_1 \geq \cdots \geq x_{k-2} \tag{7}
$$

has a solution for each $t = 0, 1, \ldots, k^2 - 4k + 4$.

Proof. Let $\mathbf{x}(t) = (x_1, x_2, \ldots, x_{k-2})$ be a solution of (7). Clearly, for each integer t with $0 \le t \le k-2$, $\mathbf{x}(t) = (t, 0, \ldots, 0)$ is a solution of (7). We suppose that $t \ge (k-2)+1$, and let

$$\rho(l) = k - 2 + 2\sum_{i=3}^{l}(k-i) \qquad (l = 3, 4, \ldots, k-1).$$

Then it is easily shown that

$$\begin{cases} \mathbf{x}(k-2+2m-1) = (k-3, m, 0, \ldots, 0) \text{ and} \\ \mathbf{x}(k-2+2m) = (k-2, m, 0, \ldots, 0) \\ \text{for each } m = 1, 2, \ldots, k-3 \text{ if } (k-2)+1 \le t \le \rho(3), \\ \mathbf{x}(\rho(3)+2m-1) = (k-3, k-3, m, 0, \ldots, 0) \text{ and} \\ \mathbf{x}(\rho(3)+2m) = (k-2, k-3, m, 0, \ldots, 0) \\ \text{for each } m = 1, 2, \ldots, k-4 \text{ if } \rho(3)+1 \le t \le \rho(4), \\ \mathbf{x}(\rho(4)+2m-1) = (k-3, k-3, k-4, m, 0, \ldots, 0) \text{ and} \\ \mathbf{x}(\rho(4)+2m) = (k-2, k-3, k-4, m, 0, \ldots, 0) \\ \text{for each } m = 1, 2, \ldots, k-5 \text{ if } \rho(4)+1 \le t \le \rho(5), \\ \cdots \\ \mathbf{x}(\rho(k-3)+2m-1) = (k-3, k-3, k-4, \ldots, k-(k-3), m, 0) \text{ and} \\ \mathbf{x}(\rho(k-3)+2m) = (k-2, k-3, k-4, \ldots, k-(k-3), m, 0) \\ \text{for each } m = 1, 2, \ldots, k-(k-2) \text{ if } \rho(k-3)+1 \le t \le \rho(k-2), \\ \mathbf{x}(\rho(k-2)+2m-1) = (k-3, k-3, k-4, \ldots, k-(k-2), m) \text{ and} \\ \mathbf{x}(\rho(k-2)+2m) = (k-2, k-3, k-4, \ldots, k-(k-2), m) \\ \text{for each } m = 1, 2, \ldots, k-(k-1) \text{ if } \rho(k-2)+1 \le t \le \rho(k-1) = k^2-4k+4 \end{cases}$$

is a solution of (7) for each $t = (k-2)+1, \ldots, k^2 - 4k + 4$. Thus the proof is complete. ■

¿From Lemma 2.1, for each $k \times k$ pattern $\mathcal{L}$ in echelon form, we know that $\mathcal{L} + \mathcal{U}$ is a fully indecomposable $k \times k$ orthogonal pattern. We take R_1 as $\mathcal{L} + \mathcal{U}$. Then $x_{k-1} = x_k = 0$. Thus from Lemma 3.3, we may take a solution $\mathbf{x}(t) = (x_1, x_2, \ldots, x_{k-2}, 0, 0)$ with the same values $x_1, x_2, \ldots, x_{k-2}$ as ones in the proof of Lemma 3.3 to the system $x_1 + 2x_2 + \cdots + 2x_k = t$ for each $t = 0, 1, \ldots, k^2 - 4k + 4$. This implies that there is a fully indecomposable $k \times k$ orthogonal pattern R_1 such that $x_1 + 2x_2 + \cdots + 2x_k = t$ for each $t = 0, 1, \ldots, k^2 - 4k + 4$.

We denote $R(t) := R_1 = \mathcal{L} + \mathcal{U}$ such that $x_1 + 2x_2 + \cdots + 2x_k = t$, and for a suitable pattern $\mathcal{L}$.

For example, let $k = 5$. Then $R(0) = [*'s]$, and

$$R(1) = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix}, \ R(2) = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix}, \ R(3) = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \\ 0 & * & * & * & * \end{bmatrix},$$

$$R(4)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ 0 & * & * & * & *\\ 0 & 0 & * & * & *\end{bmatrix},\ R(5)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ 0 & * & * & * & *\\ 0 & * & * & * & *\\ 0 & 0 & * & * & *\end{bmatrix},\ R(6)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & * & * & *\end{bmatrix},$$

$$R(7)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ 0 & * & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & * & * & *\end{bmatrix},\ R(8)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & * & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & 0 & * & *\end{bmatrix},\ R(9)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ 0 & * & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & 0 & * & *\end{bmatrix}.$$

Note that

$$R(k^2-4k+4)=\begin{bmatrix}* & * & \cdots & * & *\\ * & * & \cdots & * & *\\ 0 & * & \ddots & * & *\\ \vdots & \ddots & \ddots & * & *\\ 0 & \cdots & 0 & * & *\end{bmatrix}.$$

And also, we show that there is a fully indecomposable $k\times k$ orthogonal pattern $R(t)$ for each $t=k^2-4k+5,\dots,k^2-3k+2$. This will be done from $R(k^2-4k+4)$ by the following consecutive way:

In order to get $R(k^2-4k+5)$, permute the column 1 and column 2 of $R(k^2-4k+4)$, and to get $R(k^2-4k+6)$, permute the column 1 and column 3 of $R(k^2-4k+4)$, and so on, and finally in order to get $R(k^2-3k+2)$, we permute the column 1 and column $k-1$ of $R(k^2-4k+4)$. Then for each $t=k^2-4k+5,\dots,k^2-3k+2$, $R(t)$ is still fully indecomposable $k\times k$ orthogonal pattern.

For example, let $k=5$. Then

$$R(10)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & 0 & * & * & *\\ 0 & 0 & * & * & *\\ 0 & 0 & 0 & * & *\end{bmatrix},\ R(11)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & * & 0 & * & *\\ * & 0 & 0 & * & *\\ 0 & 0 & 0 & * & *\end{bmatrix},\ R(12)=\begin{bmatrix}* & * & * & * & *\\ * & * & * & * & *\\ * & * & * & 0 & *\\ * & 0 & * & 0 & *\\ * & 0 & 0 & 0 & *\end{bmatrix}.$$

Thus we have shown the following lemma.

LEMMA 3.4. *Let $n=2m-1$. For $k=2,3,\dots,m$, there exists a fully indecomposable $n\times n$ orthogonal pattern with p nonzero entries such that*

$$4n-4+(k^2-4k+4)\le p\le 4n-4+(2k^2-7k+6). \tag{8}$$

In (8), we should note that

$$\begin{aligned} p&=4n-4 && \text{if } k=2,\\ 4n-3\le p&\le 4n-1 && \text{if } k=3,\\ 4n\le p&\le 4n+6 && \text{if } k=4,\\ 4n+5\le p&\le 4n+17 && \text{if } k=5,\\ &\cdots\\ \frac{n^2+10n-7}{4}\le p&\le \frac{n^2+3n-2}{2} && \text{if } k=m.\end{aligned}$$

In particular, for $k-1$, since

$$4n-4+(k^2-6k+9) \le p \le 4n-4+(2k^2-11k+15)$$

we get

$$2k^2-11k+15-(k^2-4k+4)=k^2-7k+11>0 \quad \text{for} \quad k\ge 5.$$

Thus there are orthogonal patterns with the same number of nonzero entries which are overlapped between $k-1$ and k for $k \ge 5$. The following theorem is an immediate consequence of Theorem 2.2 and Lemma 3.4.

THEOREM 3.5. *For each integer p with $4n-4 \le p \le n^2$, if n is an odd number then there exists a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries.*

Now we only need to consider the even n case. If $n = 2$ or 4 then from Lemma 2.1, there exists a fully indecomposable $n \times n$ orthogonal matrix with exactly p nonzero entries such that $4n-4 \le p \le n^2$. Thus we may assume that $n = 2m \ge 6$.

For each $k = 2, \dots, m$, we define the $m \times (m+1)$ (0,1)-matrix $A'_k = [a_{ij}]$ with row sums $r_1, r_2, \dots, r_m$ and column sums $c_1, c_2, \dots, c_{m+1}$ by

$$a_{ij} = \begin{cases} 1 & \text{if } i=j;\ i=1 \text{ and } j=2,\dots,k; \\ & \quad j=i+1 \text{ for } i=k,\dots,m, \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$A'_k = \begin{bmatrix} \overbrace{1 & 1 & \cdots & \cdots & 1}^{k \text{ times}} & 0 & \cdots & 0 & 0 \\ & 1 & 0 & & & & & & \\ & & \ddots & \ddots & & & O & & \\ & & & 1 & 0 & & & & \\ & & & & 1 & 1 & & & \\ & & O & & & 1 & \ddots & & \\ & & & & & & \ddots & 1 & \\ & & & & & & & 1 & 1 \end{bmatrix}_{m\times(m+1)}. \qquad (9)$$

For $k = m+1$, define

$$A'_{m+1} = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ & 1 & & & & \\ & & \ddots & & O & \\ & O & & 1 & & \\ & & & & 1 & 0 \end{bmatrix}.$$

Thus $\#(A'_k) = \#(A'_{m+1}) = 2m$, and for $k = 2, \dots, m$,

$$r_1 = k,\ r_2 = \cdots = r_{k-1} = 1,\ r_k = \cdots = r_m = 2, c_1 = 1,\ c_2 = \cdots = c_m = 2,\ c_{m+1} = 1.$$

If we take R_i's and C_j's as full orthogonal patterns of the order r_i and c_j for each $i, j = 1, 2, \dots, m$ and $j = m+1$, then the resulting woven matrix $M(A'_k)$ has the form

$$
\left[\begin{array}{c|cc|c|cc|cc|cc|cc|c|cc|cc|c}
* & * & * & & * & * & * & * & & & & & & & & & & \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & & & & & & & & & & \\
* & * & * & & * & * & * & * & & & & & & & & & & \\
\hline
 & * & * & & & & & & & & & & & & & & & \\
\hline
 & & & \ddots & & & & & & & & & & & & & & \\
\hline
 & & & & * & * & & & & & & & & & & & & \\
\hline
 & & & & & & * & * & * & * & & & & & & & & \\
 & & & & & & * & * & * & * & & & & & & & & \\
\hline
 & & & & & & & & * & * & * & * & & & & & & \\
 & & & & & & & & * & * & * & * & & & & & & \\
\hline
 & & & & & & & & & & & & \ddots & & & & & \\
\hline
 & & & & & & & & & & & & & * & * & * & * & \\
 & & & & & & & & & & & & & * & * & * & * & \\
\hline
 & & & & & & & & & & & & & & & * & * & * \\
 & & & & & & & & & & & & & & & * & * & *
\end{array}\right].
$$

Thus

$$
\begin{aligned}
\#(M(A'_k)) &= k(2k-1) + 2(k-2) + 4(n-k-(k-2)-2) + 6 \\
&= 4n - 4 + (2k^2 - 7k + 6),
\end{aligned}
$$

and

$$
\#(M(A'_{m+1})) = (m+1)n + 2(n-m-1) = \frac{n^2+4n-4}{2}.
$$

Remark 3.6. *If $k = 2$ in (9) then $\#(M(A'_k)) = 4n - 4$ and $M(A'_2)$ is a unique orthogonal pattern for an even n as noted in [BBS].*

Now, for a $k \times k$ orthogonal pattern R_1 with a zero entry, let x_i denote the number of zero entries in the ith column of R_1. Then for full orthogonal patterns R_i $(i \neq 1)$ and C_j, we get

$$
\#(M(A'_k)) = 4n - 4 + (2k^2 - 7k + 6) - (x_1 + 2x_2 + 2x_3 + \cdots + 2x_k).
$$

If we apply the similar method which was used in the previous odd n for A'_k in (9), we can show that there is a fully indecomposable $n \times n$ orthogonal pattern with p nonzero entries for each p such that

$$
4n - 4 \leq p \leq \frac{n^2+n+4}{2}.
$$

Thus from (2), it is sufficient to show that there is a fully indecomposable $n \times n$ orthogonal pattern with p nonzero entries for each p such that

$$\frac{n^2+n+4}{2}+1 \leq p \leq \frac{n^2+3n-2}{2}-1. \tag{10}$$

In order to construct those, we shall use A'_{m+1} as a lattice of weaving. For a fully indecomposable $(m+1)\times(m+1)$ orthogonal pattern R_1 with a zero entry, let x_i denote the number of zero entries in the ith column of R_1. And we take the full orthogonal patterns R_i $(i \neq 1)$ and C_j. Then it can be easily shown that

$$\#(M(A'_{m+1})) = \frac{n^2+4n-4}{2} - (x_1+2x_2+2x_3+\cdots+2x_m+x_{m+1}).$$

For $x_1+2(x_2+\cdots+x_m)+x_{m+1}=t$, let $R(t)$ denote an $(m+1)\times(m+1)$ orthogonal pattern obtained with the similar method which was used in the previous odd n case. Then it is shown similarly that there are fully indecomposable orthogonal patterns

$$R(0), R(1), \ldots, R((m-1)^2) = R\left(\frac{n^2-4n+4}{4}\right).$$

It follows that there is a fully indecomposable $n\times n$ orthogonal pattern with p nonzero entries for each p such that

$$\frac{n^2+12n-12}{4} \leq p \leq \frac{n^2+4n-4}{2}. \tag{11}$$

From (10), (11), since

$$\frac{n^2+n+4}{2}+1-\frac{n^2+12n-12}{4} = \frac{(n-4)(n-6)}{4} \geq 0 \text{ if } n \geq 6,$$

and

$$\frac{n^2+4n-4}{2}-\frac{n^2+3n-2}{2}+1 = \frac{n-4}{2} > \text{ if } n \geq 6,$$

we have the following theorem.

THEOREM 3.7. *For each integer p with $4n-4 \leq p \leq n^2$, if n is an even number then there exists a fully indecomposable $n\times n$ orthogonal matrix with exactly p nonzero entries.*

This leads us to the existence theorem of orthogonal matrix as following.

THEOREM 3.8. *For each integer p with $4n-4 \leq p \leq n^2$, there exists a fully indecomposable $n\times n$ orthogonal matrix with exactly p nonzero entries.*

Proof. This follows from Theorem 3.5 and 3.7. ∎

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