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# PARAMETER ESTIMATION PROBLEM FOR NONHYSTERETIC INFILTRATION IN SOIL

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**Abstract.** Nonhysteretic infiltration in nonswelling soil is modelled by the Burgers equation under appropriate physical conditions. For this nonlinear partial differential equation the modal approximation scheme is used for estimating parameters such as soil water diffusivity and hydraulic conductivity. The parameter estimation convergence is proved, and numerical experiments are performed.

## 0. Introduction.

The Burgers equation introduced by Burgers [6,7] as a simple mathematical model for turbulence has been studied by many researchers in various areas such as fluid flows, traffic flows, geophysics, and soil sciences, etc. [2,4,8,12,13,14,16,17,19,20].

In this paper, we consider the Burgers equation in the context of estimating geophysical parameters that control the fate and movement of rainfall infiltration in soil. Movement of water and soluble contaminants such as pestcides or herbicides in subsurface has been an important issue in soil sciences, crop/plant farms, and environmental concerns. To predict movement of the solution in soil, geophysical parameters such as water diffusivity and hydraulic conductivity have to be estimated accurately from field sample data.

Flow in subsurface unsaturated zone is usually modelled by the Richards equation or the Fokker-Planck diffusion-convection equation which is nonlinear in nature. These equations are derived by combining the mass conservation equation and Darcy's law, assuming that air effects and compressibility of both water and solid matrix are negligible [5,12,24]. For nonhysteretic infiltration in nonswelling soil, when the soil water diffusivity is constant and the unsaturated hydraulic conductivity is proportional to

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the square of the reduced water content, the Richards equation becomes the Burgers equation [12]. In this paper we consider a parameter estimation problem for the following form of Burgers equation:

$$\frac{\partial \vartheta}{\partial t} = \delta \frac{\partial^2 \vartheta}{\partial z^2} - 2a(\vartheta + b)\frac{\partial \vartheta}{\partial z} \tag{0.1}$$

with the initial and boundary conditions

$$\vartheta(z,0) = \vartheta_0, \quad \left[a(\vartheta+b)^2 - \delta\frac{\partial \vartheta}{\partial z}\right](0,t) = \mathrm{F}_0, \quad \vartheta(\mathrm{L},t) = \vartheta_{\mathrm{L}}. \tag{0.2}$$

In (0.1)-(0.2), $\vartheta$ is the volumetric water content, $t$ is the time, $z$ is the depth, $\mathrm{F}_0$, $\vartheta_0$ and $\vartheta_{\mathrm{L}}$ are given constants, and $\delta$, $a$ and $b$ are parameters to be estimated. The parameter $\delta$ is the soil water diffusivity and the quantity $a(\vartheta+b)^2$ describes the hydraulic conductivity. Note that equation (0.1) and the flux boundary condition at $z = 0$ are nonlinear. The model (0.1)-(0.2) describes the constant rate rainfall infiltration in the soil of finite depth [16]. Here, all parameters and constants in our model (0.1)-(0.2) are assumed to be chosen so that the infiltration rates are less than the saturated hydraulic conductivity, and hence the soil remains unsaturated. Finally, we assume the following compatibility condition

$$\vartheta_0 = \vartheta_{\mathrm{L}} \tag{0.3}$$

holds, which implies that the initial data satisfies the given boundary condition. For more discussions of related models, see [12,15,16,22].

For our parameter estimation problem, we will denote the parameter set by $Q = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- = \{(\delta, a, b)\}$, where $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the set of positive and negative real numbers, respectively. The choice of this parameter set comes from the geophysical consideration. The solution for equations (0.1)-(0.2) with a parameter $q \in Q$ will be denoted by $\vartheta(z,t;q)$. Then the parameter estimation problem is to determine the parameters from observations.

Let $\mathrm{T}_o$ be a fixed observation time, $\{z_i\}_{i=1}^m$ a fixed set of observation points in $[0,\mathrm{L}]$, and $\Delta z$ a sufficiently small fixed positive number. Then we define a parameter-to-output mapping $\Phi : Q \to \mathbb{R}^m$ as $\Phi(q) = (\vartheta_1(q), \dots, \vartheta_m(q))$, where

$$\vartheta_i(q) = \frac{1}{\Delta z}\left[\int_{z_i - \frac{1}{2}\Delta z}^{z_i + \frac{1}{2}\Delta z} \vartheta(z, \mathrm{T}_o; q)\, dz\right]. \tag{0.4}$$

Notice that each $\vartheta_i$ represents the averaged water contents in a small neighborhood of $z_i$ at time $\mathrm{T}_o$. This form of observation is chosen from the consideration that each measurement represents an averaged water contents resided in the soil surrounding $z_i$ rather than point observation, i.e., water contents measured at the point itself. Then the inverse problem is to find the inverse mapping of $\Phi$. Due to uncertain disturbances in modelling and measurements, it is naturally suggested to consider the following optimization problem.

**Problem (P).** Let $\tilde{Q}$ be an admissible parameter subset of the parameter space $Q$. Given a set of measurements $\theta = (\theta_1, \dots, \theta_m)$, find $q^* \in \tilde{Q}$ that minimizes the cost functional $\Gamma : \tilde{Q} \to \mathbb{R}$ defined by

$$\Gamma(q) := \|\Phi(q) - \theta\|_2^2 = \sum_{i=1}^{m} [\vartheta_i(q) - \theta_i]^2 ,$$

where $\vartheta_i(q)$ is defined in (0.4).

Note that the cost functional $\Gamma$ is defined via the solution of a partial differential equation. So, our parameter estimation problem is an infinite dimensional one, and hence, we need to approximate it by a sequence of finite dimensional problems. The question regarding convergence of the sequence which is called the Parameter Estimation Convergence (PEC) is to be answered. Many parameter estimation problems have been studied for models arising in groundwater flow (see, e.g., [1,3,25] and the references cited there). In this paper, we propose an approximation scheme (modal scheme) for the problem (P) and prove the convergence result of the proposed scheme. To treat the nonlinear term $\vartheta\vartheta_z$ and the nonlinearity of the boundary conditions, the system (0.1)–(0.2) will be converted into a linear equation with linear homogeneous boundary conditions by using appropriate change of variables. However, as a result of this transformation the observation operator becomes more complex and nonlinear.

The numerical result of this work had been presented in [9]. In [10,11], the authors considered the related parameter estimation problems, where the initial water distribution and the flux history are estimated under the assumption that all the geophysical parameters are assumed to be known.

This paper is organized as follows. In Section 1, we describe a transformation which leads the system (0.1)-(0.2) to a linear equation with homogenous boundary conditions, and we formulate our problem via the transformed variables. A finite dimensional approximation scheme is proposed and the parameter estimation convergence of the scheme is proved in Section 2. Some numerical simulation results for our parameter estimation scheme are presented in Section 3, and concluding remarks are made in Section 4.

## 1. Transformation.

The Burgers equation (0.1) can be transformed to a linear equation by the Cole-Hopf transformation [13,16,17] under the compatibility conditions and the smoothness of the solution. To change the nonhomogeneous boundary condition to a homogeneous one, we take

$$v(z,t) = -h(t;q)g(z;q) + e^{\frac{a}{\delta}[\mathrm{F}_0 t - bz - \int_0^z \vartheta(x,t)\,dx]}, \tag{1.1}$$

where $h(t;q) = e^{a\mathrm{F}_0 t/\delta}$ and $g(z;q) = 1 - [\alpha(q)z]/[1 + \alpha(q)\mathrm{L}]$, and $\alpha(q) = a(\vartheta_\mathrm{L} + b)/\delta$. Then (0.1)-(0.2) are transformed to the following equation with homogeneous boundary conditions :

$$\frac{\partial v}{\partial t} = \delta \frac{\partial^2 v}{\partial z^2} - h'(t;q)g(z;q), \tag{1.2}$$

$$\begin{gathered} v(z,0) = e^{-\gamma(q)z} - g(z;q), \\ v(0,t) = 0, \quad \left[\frac{\partial v}{\partial z} + \alpha(q)v\right](\mathrm{L},t) = 0, \end{gathered} \tag{1.3}$$

where $\gamma(q) = a(\vartheta_0 + b)/\delta$. Note that $\alpha(q) = \gamma(q)$ by the compatibility condition (0.3).

From now on we fix a finite time $\mathrm{T} \geq \mathrm{T}_o$ and consider the problem (1.2)–(1.3) in the region $[0,\mathrm{L}] \times [0,\mathrm{T}]$. It is well known from the linear semigroup theory that the problem (1.2)–(1.3) has a unique solution $v(z,t;q) \in C([0,\mathrm{T}];L^2(0,\mathrm{L}))$, which satisfies $v(\cdot,t;q) \in \mathcal{S}$ for each $t \in [0,\mathrm{T}]$, where

$$\mathcal{S} = \{\zeta \in C^1[0,\mathrm{L}] \,|\, \zeta(0) = 0,\ [\zeta' + \alpha(q)\zeta]\,(\mathrm{L}) = 0\} \tag{1.4}$$

(see [21]). In particular, the solution is continuously differentiable with respect to the space variable. So, once we obtain the solution $v(z,t;q)$ of (1.2)-(1.3), the volumetric water content $\vartheta(z,t;q)$ can be recovered by the transformation

$$\vartheta(z,t;q) = -\frac{\delta}{a}\left[\frac{h(t;q)g'(z;q) + v_z(z,t;q)}{h(t;q)g(z;q) + v(z,t;q)}\right] - b. \tag{1.5}$$

On the other hand, we note from (1.1) that the parameter-to-output mapping $\Phi : Q \to \mathbb{R}^m$ is represented as $\Phi(q) = (v_1(q), \ldots, v_m(q))$, where

$$v_i(q) = -\frac{1}{\Delta z}\frac{\delta}{a}\ln\left[\frac{v(z_i + \frac{1}{2}\Delta z, \mathrm{T}_o; q) + g(z_i + \frac{1}{2}\Delta z; q)h(\mathrm{T}_o; q)}{v(z_i - \frac{1}{2}\Delta z, \mathrm{T}_o; q) + g(z_i - \frac{1}{2}\Delta z; q)h(\mathrm{T}_o; q)}\right] - b, \tag{1.6}$$

and, hence, our inverse problem can be stated as

**Problem (P).** Let $\tilde{Q}$ be an admissible parameter subset of the parameter space $Q$. Given a set of measurements $\theta = (\theta_1, \ldots, \theta_m)$, find $q^* \in \tilde{Q}$ that minimizes

$$\Gamma(q) := \|\Phi(q) - \theta\|_2^2 = \sum_{i=1}^{m}[v_i(q) - \theta_i]^2 \quad \text{in } \tilde{Q},$$

where $v_i(q)$ is given by (1.6).

To obtain the solution of (1.2)–(1.3) in analytical form we construct a positive increasing sequence $\{\lambda_n(q)\}$ by solving

$$\lambda_n \cos\lambda_n\mathrm{L} + \alpha(q)\sin\lambda_n\mathrm{L} = 0 \tag{1.7}$$

and let

$$\mathrm{J}_n(q) = 2\left[\frac{\lambda_n^2(q)+\alpha^2(q)}{[\lambda_n^2(q)+\alpha^2(q)]\mathrm{L}+\alpha(q)}\right].$$

Define a sequence $\{\zeta_n(z;q)\}$ of functions by

$$\zeta_n(z;q) = \sqrt{\mathrm{J}_n(q)}\sin\lambda_n(q)z.$$

Then $\{\zeta_n(z;q)\}$ forms a complete orthonormal set in $L^2(0,\mathrm{L})$ and each element $\zeta_n(z;q)$ belongs to $\mathcal{S}$, where $\mathcal{S}$ is given by (1.4). Thus, the solution $v(z,t;q)$ of (1.2)–(1.3) can be represented by

$$v(z,t;q) = \sum_{n=1}^{\infty}\omega_n(t;q)\zeta_n(z;q), \tag{1.8}$$

where the coefficients $\omega_n(t;q)$ are to be determined. By substuting $v(z,t;q)$ in (1.8) into (1.2)–(1.3), we obtain the following initial value problems for the coefficients $\{\omega_n(t;q)\}$ :

$$\begin{aligned}\omega_n' &= -\delta\lambda_n^2(q)\omega_n - h'(t;q)\left[\int_0^{\mathrm{L}} g(z;q)\zeta_n(z;q)\,dz\right],\\ \omega_n(0) &= \int_0^{\mathrm{L}}\left[e^{-\gamma z}-g(z;q)\right]\zeta_n(z;q)\,dz.\end{aligned} \tag{1.9}$$

It is easy to see that the solution of (1.9) is given by

$$\begin{aligned}\omega_n(t;q) = &-\left\{\frac{\sqrt{\mathrm{J}_n(q)}e^{-\delta\lambda_n^2(q)t}}{\lambda_n(q)[(a/\delta)\mathrm{F}_0+\delta\lambda_n^2(q)]}\right\}\left\{\frac{a}{\delta}\mathrm{F}_0 e^{(a/\delta)\mathrm{F}_0 t+\delta\lambda_n^2(q)t}+\right.\\ &\left.+\left[\frac{(\delta\alpha^2(q)-(a/\delta)\mathrm{F}_0)\lambda_n^2(q)}{\lambda_n^2(q)+\alpha^2(q)}\right]\right\}.\end{aligned}$$

Thus, $v(z,t;q)$ with coefficients above is the unique solution of (1.2)–(1.3), i.e., the series (1.8) converges uniformly on $[0,\mathrm{L}]\times[0,\mathrm{T}]$. More specifically, it is easy to show that for each $n\in\mathbb{N}$,

$$\lambda_n(q)\in\left(\frac{(2n-1)\pi}{2\mathrm{L}},\frac{n}{\mathrm{L}}\pi\right);\quad |\mathrm{J}_n(q)|\le\frac{2}{\mathrm{L}}, \tag{1.10}$$

and, hence, for all $(z,t)\in[0,\mathrm{L}]\times[0,\mathrm{T}]$,

$$\left|\omega_n(t;q)\sqrt{\mathrm{J}_n(q)}\sin\lambda_n(q)z\right|\le\frac{\mathrm{C}}{n^3}$$

for some constant C independent of $n$. Therefore, the infinite series in (1.8) converges uniformly on $[0,\mathrm{L}]\times[0,\mathrm{T}]$. It is clear that $v(z,t;q)$ in (1.8) satisfies the initial and boundary conditions (1.3). Moreover, it is easy to see that for all $(z,t)\in[0,\mathrm{L}]\times[0,\mathrm{T}]$,

$$\left|\omega_n(t;q)\sqrt{\mathrm{J}_n(q)}\lambda_n(q)\cos\lambda_n(q)z\right|\le\frac{\mathrm{C}}{n^2}$$

for some constant C independent of $n$. Thus, the spatial derivative $v_z$ of (1.8) also exists as a continuous function on $[0,\mathrm{L}]\times[0,\mathrm{T}]$.

## 2. Parameter estimation scheme.

We take a compact subset $\tilde{Q}$ of the parameter space $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^-$ as an admissible parameter subset in Problem (P). Note that Problem (P) is infinite dimensional one since the solution is in the infinite dimensional space. Thus, we need to approximate Problem (P) by a sequence of finite dimensional ones.

For each $N \in \mathbb{N}$ and $q \in \tilde{Q}$, define $v^N(z,t;q)$ as the following finite sum

$$v^N(z,t;q) = \sum_{n=1}^{N} \omega_n(t;q)\sqrt{\mathrm{J}_n(q)}\sin\lambda_n(q)z. \tag{2.1}$$

Then, $v^N(z,t;q)$ is the finite dimensional solution of the problem (1.2)–(1.3) in the subspace generated by $\{\zeta_n(z;q)\}_{1\le n\le N}$. It is clear that for each fixed $q \in \tilde{Q}$, the sequence $\{v^N(z,t;q)\}$ converges uniformly to the solution $v(z,t;q)$ on $[0,\mathrm{L}]\times[0,\mathrm{T}]$, and the corresponding finite dimensional parameter estimation problem becomes :

**Problem ($\mathbf{P}^N$).** Given a set of measurements $\theta = (\theta_1,\dots,\theta_m)$, find $q^* \in \tilde{Q}$ that minimizes

$$\Gamma^N(q) = \|\Phi^N(q)-\theta\|_2^2 = \sum_{i=1}^{m}\left[v_i^N(q)-\theta_i\right]^2 \quad \text{in } \tilde{Q},$$

where $\Phi^N(q) = \big(v_1^N(q),\dots,v_m^N(q)\big)$, and

$$v_i^N(q) = -\frac{1}{\Delta z}\frac{\delta}{a}\ln\left[\frac{v^N(z_i+\frac{1}{2}\Delta z,\mathrm{T}_o;q)+g(z_i+\frac{1}{2}\Delta z;q)h(\mathrm{T}_o;q)}{v^N(z_i-\frac{1}{2}\Delta z,\mathrm{T}_o;q)+g(z_i-\frac{1}{2}\Delta z;q)h(\mathrm{T}_o;q)}\right]-b.$$

In the rest of this section, we will show that each problem ($\mathrm{P}^N$) has a solution $q^N$ and that the sequence $\{q^N\}$ has a convergent subsequence which converges to a solution of the original problem (P). A parameter estimation scheme having this property is called a *parameter estimation convergent* (PEC) scheme [3,Chapter 3]. These properties will be accomplished by showing that the true solution $v(z,t;q)$ as well as its approximations $v^N(z,t;q)$ are continuous with respect to the parameter $q$. First we prove

**Lemma 2.1.** *If $q=(\delta,a,b), \tilde{q}=(\tilde{\delta},\tilde{a},\tilde{b}) \in \tilde{Q}$, then, for each $n \in \mathbb{N}$, we have*

$$|\lambda_n(q)-\lambda_n(\tilde{q})| \le \frac{\pi}{\mathrm{L}^2}\Big|\frac{1}{\alpha(q)}-\frac{1}{\alpha(\tilde{q})}\Big| = \frac{\pi}{\mathrm{L}^2}\Big|\frac{\delta}{a(\vartheta_\mathrm{L}+b)}-\frac{\tilde{\delta}}{\tilde{a}(\vartheta_\mathrm{L}+\tilde{b})}\Big|. \tag{2.2}$$

*Proof.* First, notice that if $\alpha(q) > \alpha(\tilde{q})$, then $\lambda_n(q) > \lambda_n(\tilde{q})$ and $\lambda_n(q)-\lambda_n(\tilde{q}) < \lambda_{n-1}(q)-\lambda_{n-1}(\tilde{q})$ for all $n \in \mathbb{N}$. Thus, it suffices to show the inequality (2.2) for $n=1$. From (1.7) we know that

$$-\lambda_1(q)/\alpha(q) = \tan\mathrm{L}\lambda_1(q),\ \ -\lambda_1(\tilde{q})/\alpha(\tilde{q}) = \tan\mathrm{L}\lambda_1(\tilde{q}).$$

By the Mean Value Theorem,

$$\frac{\lambda_1(\tilde{q})/\alpha(\tilde{q}) - \lambda_1(q)/\alpha(q)}{\lambda_1(q) - \lambda_1(\tilde{q})} = \frac{\tan \mathrm{L}\lambda_1(q) - \tan \mathrm{L}\lambda_1(\tilde{q})}{\lambda_1(q) - \lambda_1(\tilde{q})} = \frac{\mathrm{L}}{\cos^2 \mathrm{L}\xi} \geq \mathrm{L}$$

for some $\xi \in (\lambda_1(\tilde{q}), \lambda_1(q))$. Therefore, by (1.10), we obtain

$$\begin{aligned}\lambda_1(q) - \lambda_1(\tilde{q}) &\leq \frac{1}{\mathrm{L}}\left|\frac{\lambda_1(\tilde{q})}{\alpha(\tilde{q})} - \frac{\lambda_1(q)}{\alpha(q)}\right| \leq \frac{\lambda_1(\tilde{q})}{\mathrm{L}}\left|\frac{1}{\alpha(\tilde{q})} - \frac{1}{\alpha(q)}\right| \\ &\leq \frac{\pi}{\mathrm{L}^2}\left|\frac{1}{\alpha(\tilde{q})} - \frac{1}{\alpha(q)}\right|.\end{aligned}$$

The proof is completed. □

**Lemma 2.2.** *Suppose $\{q^m\}$ is a sequence in $\tilde{Q}$ and $q^m \to q^0$ in $\tilde{Q}$. Then, for each $N \in \mathbb{N}$,*

$$|v^N(z,t;q^m) - v^N(z,t;q^0)| \to 0 \quad \textit{as } m \to \infty,$$

*uniformly in $z \in [0, \mathrm{L}]$ and in $t \in [0, \mathrm{T}]$.*

*Proof.* For each $n$, by Lemma 2.1, we have

$$|\sin \lambda_n(q^m)z - \sin \lambda_n(q^0)z| \leq |\lambda_n(q^m) - \lambda_n(q^0)|\mathrm{L} \leq \frac{\pi}{\mathrm{L}}\left|\frac{1}{\alpha(q^m)} - \frac{1}{\alpha(q^0)}\right|,$$

for all $z \in [0, \mathrm{L}]$. From Lemma 2.1 and the expression of $\omega_n(t;q)$, we have for each $n$,

$$\left|\sqrt{\mathrm{J}_n(q^m)}\omega_n(t;q^m) - \sqrt{\mathrm{J}_n(q^0)}\omega_n(t;q^0)\right| \to 0 \quad \text{as } m \to \infty,$$

uniformly in $t \in [0, \mathrm{T}]$. Observing the expression (2.1) of $v^N$ the lemma follows these estimates. □

**Lemma 2.3.** *Suppose $\{q^m\}$ is a sequence in $\tilde{Q}$ and $q^m \to q^0$ in $\tilde{Q}$. Then*

$$|v(z,t;q^m) - v(z,t;q^0)| \to 0 \quad \textit{as } m \to \infty,$$

*uniformly in $z \in [0, \mathrm{L}]$ and in $t \in [0, \mathrm{T}]$.*

*Proof.* First, from the compactness of $\tilde{Q}$ and the estimate (1.10) we see that

$$|\omega_n(t;q)\zeta_n(z;q)| \leq \frac{\mathrm{C}}{\lambda_n^3(q)} \leq \frac{\mathrm{C}}{n^3}$$

for some constant C depending only on $\tilde{Q}$. So, for any given $\varepsilon > 0$, there exists a number $\mathrm{M}_1 = \mathrm{M}_1(\varepsilon)$ such that

$$\sum_{n=\mathrm{M}_1}^{\infty} |\omega_n(t;q)\zeta_n(z;q)| \leq \frac{\varepsilon}{4} \tag{2.3}$$

for all $z \in [0, \mathrm{L}]$, $t \in [0, \mathrm{T}]$, and $q \in \tilde{Q}$. Note that each $\omega_n(t; q)\zeta_n(z; q)$ is uniformly continuous in $z \in [0, \mathrm{L}]$, $t \in [0, \mathrm{T}]$, and $q \in \tilde{Q}$. Therefore, for the given $\varepsilon > 0$, we can find $\mathrm{M}_2 = \mathrm{M}_2(\varepsilon)$ such that for any $m \geq \mathrm{M}_2$,

$$|\omega_n(t; q^m)\zeta_n(z; q^m) - \omega_n(t; q^0)\zeta_n(z; q^0)| \leq \frac{\varepsilon}{2\mathrm{M}_1} \tag{2.4}$$

for all $z \in [0, \mathrm{L}]$, $t \in [0, \mathrm{T}]$, and for all $n = 1, \dots, \mathrm{M}_1$. Thus,

$$\begin{aligned} &|v^{\mathrm{M}_1}(z, t; q^m) - v^{\mathrm{M}_1}(z, t; q^0)| \\ &\quad \leq \sum_{n=1}^{\mathrm{M}_1} |\omega_n(t; q^m)\zeta_n(z; q^m) - \omega_n(t; q^0)\zeta_n(z; q^0)| \\ &\quad \leq \frac{\varepsilon}{2} \end{aligned} \tag{2.5}$$

for all $z \in [0, \mathrm{L}]$ and for all $t \in [0, \mathrm{T}]$, whenever $m \geq \mathrm{M}_2$. Thus, if $m \geq \mathrm{M}_2$, from (2.3) and (2.5), we have for all $z \in [0, \mathrm{L}]$ and for all $t \in [0, \mathrm{T}]$,

$$\begin{aligned} |v(z, t; q^m) - v(z, t; q^0)| &\leq |v^{\mathrm{M}_1}(z, t; q^m) - v^{\mathrm{M}_1}(z, t; q^0)| \\ &\quad + \sum_{n=\mathrm{M}_1}^{\infty} |\omega_n(t; q^0)\zeta_n(z; q^0)| \\ &\quad + \sum_{n=\mathrm{M}_1}^{\infty} |\omega_n(t; q^m)\zeta_n(z; q^m)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \end{aligned}$$

This completes the proof. □

From Lemma 2.2 and Lemma 2.3 we obtain

**Lemma 2.4.** *Suppose $\{q^N\}$ is a sequence in $\tilde{Q}$ and $q^N \to q^0$ in $\tilde{Q}$. Then*

$$|v^N(z, t; q^N) - v(z, t; q^0)| \to 0 \quad \text{as } N \to \infty,$$

*uniformly in $z \in [0, \mathrm{L}]$ and in $t \in [0, \mathrm{T}]$.*

It is easy to see that the parameter dependent functions $g$ and $h$ satisfy the similar estimates as in Lemma 2.2. That is, if $q^m \to q^0$ in $\tilde{Q}$, we have

$$\begin{aligned} |g(z; q^m) - g(z; q^0)| &\to 0, \\ |h(t; q^m) - h(t; q^0)| &\to 0, \end{aligned} \tag{2.6}$$

uniformly in $z \in [0, \mathrm{L}]$ and in $t \in [0, \mathrm{T}]$, as $m$ goes to infinity. Then, it is clear from the definitions of $\Gamma$ and $\Gamma^N$, Lemma 2.2, Lemma 2.3 and the estimate (2.6) that $\Gamma$ and

$\Gamma^N$ are continuous. Thus, on a compact subset $\tilde{Q}$ of $Q$, the problems ($\mathrm{P}^N$) and (P) always have solutions.

Now let $\{q^N\}$ be a sequence in $\tilde{Q}$ such that each $q^N$ is a solution of ($\mathrm{P}^N$). Since $\tilde{Q}$ is compact, there exists a convergent subsequence. Suppose $\{q^{N_k}\}$ is such a convergent subsequence and let $q^{N_k} \to q^*$ as $k$ goes to infinity. Then, by Lemma 2.4, we get

$$|v^{N_k}(z,t;q^{N_k}) - v(z,t;q^*)| \to 0$$

uniformly in $z \in [0,\mathrm{L}]$ and in $t \in [0,\mathrm{T}]$, and hence we obtain

$$|\Gamma^{N_k}(q^{N_k}) - \Gamma(q^*)| \to 0. \tag{2.7}$$

But, since each $q^{N_k}$ is a solution of ($\mathrm{P}^{N_k}$), we know that

$$\Gamma^{N_k}(q^{N_k}) \le \Gamma^{N_k}(\tilde{q}) \quad \text{for all } \tilde{q} \in \tilde{Q}. \tag{2.8}$$

Sending $k$ to infinity in (2.8) we conclude that $\Gamma(q^*) \le \Gamma(\tilde{q})$ for all $\tilde{q} \in \tilde{Q}$, which states that $q^*$ is a solution of Problem (P).

Consequently we have proved the following theorem.

**Theorem 2.5.** *($\mathrm{P}^N$) is parameter estimation convergent for the problem (P).*

In conclusion, if we are given a set of measurements and an admissible parameter set, we can construct a sequence $\{q^N\}$ by solving the problem ($\mathrm{P}^N$), and then, by finding subsequential limits of $\{q^N\}$, we can obtain a solution to the problem (P).

## 3. Numerical results.

To illustrate the parameter estimation convergence we present an example. This example was taken from [16]. All the numerical calculations in this paper were performed on a SUN SPARC-20 workstation under the MATLAB environment.

Set

$$\mathrm{L} = 25\,\mathrm{cm}\,,\ \vartheta_0 = 0.03\,,\ \vartheta_{\mathrm{L}} = 0.03\,,\ \mathrm{F}_0 = 2.04 \times 10^{-4}\,\mathrm{cm/min}$$

and assume that the true parameters are given by

$$\delta = 0.2106\,\mathrm{cm}^2/\mathrm{min}\,,\ a = 0.5928\,\mathrm{cm/min}\,,\ b = -0.0065\,\mathrm{cm}^3/\mathrm{cm}^3.$$

These soil properties are similar to those used in [12]. The parameters were chosen so that they satisfy the compatibility conditions for the system (1.2)–(1.3), and they are related to nonponding infiltration with fluxes less than the saturated hydraulic conductivity. Thus, the soil remains unsaturated. The observation time $\mathrm{T}_o = 120$ minutes and the ten observation points $z_i = \mathrm{L}/20 + (i-1)\mathrm{L}/10$, $i = 1, \dots, 10$, were chosen and $\Delta z$ was chosen sufficiently small so that the volumetric water contents were assumed to be measured approximately at these ten points. Measured water contents at the obsevation points were given by

$$\theta_i = -\frac{\delta}{a}\left[\frac{g'(z_i)h(\mathrm{T}_o) + \tilde{v}_z(z_i, \mathrm{T}_o)}{g(z_i)h(\mathrm{T}_o) + \tilde{v}(z_i, \mathrm{T}_o)}\right] - b, \quad 1 \le i \le 10,$$

where the approximation $\tilde{v}$ of $v$ was calculated by arbitrarily truncating the infinite series (1.8) for the true solution $v$ at 1500 terms. Here, we did not attempt to optimize the number of terms, since the evaluations require only a half second.

Table 1 shows the parameter estimation convergence (PEC) property for the modal scheme. The OLS-Error in Table 1 indicates the output least squared error $\|\Phi^N(q^N) - \theta\|_2^2 = \sum_{i=1}^{10}\left[v_i^N(q^N) - \theta_i\right]^2$. We started with the following initial guesses

$$\delta_0 = 1.0\,\mathrm{cm}^2/\mathrm{min}\,,\ a_0 = 1.0\,\mathrm{cm}/\mathrm{min}\,,\ b_0 = -10^{-6}\,\mathrm{cm}^3/\mathrm{cm}^3.$$

Table 1. PEC of Modal Scheme

| N | $\delta$ | $a$ | $b$ | OLS-Error |
|---|---|---|---|---|
| Initial Guess | 1.0000e+0 | 1.0000e+0 | −1.0000e−6 | |
| 20 | 3.4004e−1 | 7.2394e−1 | −3.0000e−2 | 1.8091e−3 |
| 40 | 2.7236e−1 | 7.7119e−1 | −3.0000e−2 | 2.2008e−3 |
| 80 | 2.2341e−1 | 6.3006e−1 | −1.2261e−2 | 2.5512e−4 |
| 160 | 2.1247e−1 | 5.9806e−1 | −7.3459e−3 | 4.7164e−5 |
| 320 | 2.1084e−1 | 5.9348e−1 | −6.6096e−3 | 6.3273e−6 |
| 640 | 2.1063e−1 | 5.9289e−1 | −6.5147e−3 | 8.5111e−7 |
| 1280 | 2.1061e−1 | 5.9282e−1 | −6.5028e−3 | 1.6024e−7 |
| True Value | 2.1060e−1 | 5.9280e−1 | −6.5000e−3 | |

Newton's method was used to obtain eigenvalues $\{\lambda_n\}$ in (1.7) with the stopping criterion $|\lambda_n \cos\lambda_n \mathrm{L} + \alpha \sin\lambda_n \mathrm{L}| < 10^{-8}$. To estimate parameters, the Finite-Difference-Levenberg-Marquardt method [18,23] was used. This method has been commonly used for minimization problems with least squared error functional. In Figure 1, the curves represent the solution $\vartheta^{1500}$ with the true parameters (- - - -) and the approximation $\vartheta^N$ with the estimated parameters (——) listed in Table 1. The symbol + represents the observed water contents.

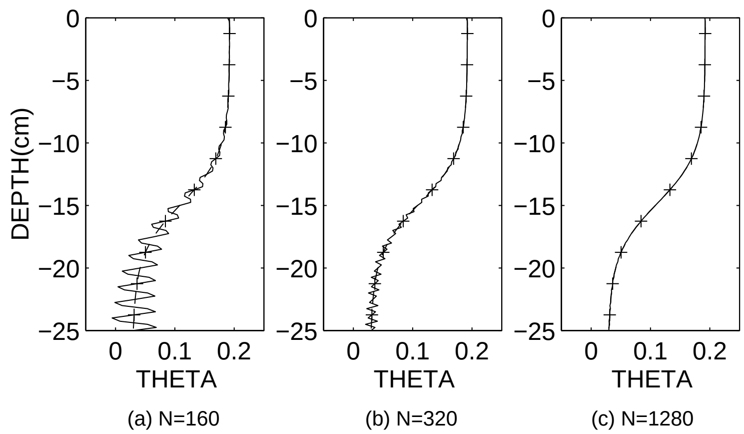


Figure 1. Modal solutions and observation data
(+: observation data, - - -: solution with true parameters,
——: approximate solution with estimated parameters)

## 4. Conclusions.

The Burgers equation is considered as a one dimensional model for vertical nonhysteretic infiltration in nonswelling soil with finite depth. We developed a systematic approximation scheme for estimating parameters such as soil water diffusivity and conductivity. The main idea dealing with nonlinearity is to change the equation into a linear form using an appropriate transformation. Numerical experiments support the theory regarding the parameter estimation convergence of the modal approximation scheme.

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