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# Legendre Tau Method for the 2-D Stokes Problem

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**Abstract**

A Legendre spectral tau approximation scheme for solving the two-dimensional stationary incompressible Stokes equations is considered. Based on the vorticity-stream function formulation and variational forms, boundary value and normal derivative of vorticity are computed. A factorization technique for matrix stems based on the Schur decomposition is derived. Several numerical experiments are performed.

## 1 Introduction

The two-dimensional stationary Stokes equations describing the motion of an incompressible fluid in a bounded domain $\Omega \subset \mathbf{R}^2$ with the boundary $\Gamma$ can be written, in terms of primitive variables, as

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $\mathbf{u}$ is the velocity, $p$ is the pressure, $\mathbf{f}$ is a field of given body forces, $\nu$ is the kinematic viscosity of the fluid, and $\mathbf{g}$ is a given field defined on $\Gamma$ satisfying the global conservation property:

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n}\, ds = 0,$$

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where $\mathbf{n}$ is the unit outer normal vector on $\Gamma$. We assume, in this paper, that $\Omega = (-1,1)\times(-1,1)$ and $\Gamma_i, i = 1,2,3,4,$ are edges of the boundary $\Gamma$, starting from the south and turning counterclockwise (see Figure 1).

By applying the curl operation, the Stokes system (1.1), in terms of vorticity $\omega$ and stream function $\psi$, becomes

$$\begin{aligned} -\nu\Delta\omega &= \mathrm{curl}\mathbf{f} \ \text{ in } \Omega, \\ -\Delta\psi &= \omega \ \text{ in } \Omega. \end{aligned} \tag{1.2}$$

Since $\psi$ is unique up to a constant, the following boundary conditions corresponding to those of (1.1) are considered: for $i = 1,2,3,4,$

$$\begin{aligned} \psi(x) &= h_i := \int_{\widehat{x_0 x}} \mathbf{g}\cdot\mathbf{n}\, ds \ \text{ on } \Gamma_i, \\ \frac{\partial\psi}{\partial\mathbf{n}} &= g_i := -\mathbf{g}\cdot\mathbf{s} \ \text{ on } \Gamma_i, \end{aligned} \tag{1.3}$$

where $\mathbf{s}$ is the tangential vector on $\Gamma$, $\psi(x_0) = 0$ for some $x_0 \in \Gamma$ and $\widehat{x_0 x}$ is the path from $x_0$ to $x$ along $\Gamma$.

The advantage of the vorticity-stream function formulation (1.2)-(1.3) is that we do not need to deal with the divergence free condition $\nabla\cdot\mathbf{u} = 0$ and the pressure $p$. Note that the divergence condition is automatically satisfied and the pressure is dropped in (1.2). These lead a low cost discretization in numerical implementation. Also, the velocity and the pressure can be easily recovered from the stream function. However, a critical drawback of the formulation (1.2)-(1.3) is the lack of boundary conditions on the vorticity $\omega$ while there are two boundary conditions on the stream function $\psi$. A well-known way to overcome this difficulty in finite difference or finite element methods is to define the boundary conditions of vorticity from the relation $\omega = -\Delta\psi$. In this paper, we derive an efficient method for finding the traces of the vorticity based on variational forms, Green's formula and the Schur decomposition through a Legendre tau approximation. The ideas are similar to those proposed by Glowinski and Pironneau[8]. However, we only deal with a sparse, symmetric matrix system in which each column of the governing matrix is obtained by solving one Laplace equation through a Legendre tau approximation(see Section 3) instead of solving a full matrix system in which each column of the corresponding matrix is computed by solving two Laplace equations as in [8].

In recent years, a number of algorithms using spectral methods have been implemented for solving the Stokes and the Navier-Stokes equations. Meanwhile, various theoretical and numerical results dealing with spectral Galerkin and spectral collocation methods have been established(see, e.g., [1-5, 13-14] and references therein). However, to the authors' knowledge, the spectral tau methods seem to be less studied, although they are frequently used in practice because of their efficiencies in solving, for example, Helmholtz type equations(see, e.g., [10] for the fast Helmholtz solver and [9] for a Chebychev tau solver). Legendre tau formulations for the Stokes problem can be founded in [12] and [15] in which the formulations are based on the velocity-pressure form with homogeneous boundary conditions.

An outline of this work is as follows. In Section 2, we introduce a decomposition of the system (1.2)-(1.3) and abstract variational forms. A Legendre tau approximation scheme and a factorization method are given in Section 3, and numerical results are presented in Section 4.

Throughout this paper, $H^s(\Omega)$ is the standard Sobolev space with the standard norm $\|\cdot\|_{H^s(\Omega)}$. We shall denote by $(\cdot,\cdot)$ the usual inner product of $L^2(\Omega)$. For any Banach (or Hilbert) spaces $X$ and $Y$, let $\mathcal{L}\ (U,V)$ be the space of all bounded linear operators from $U$ to $V$ and let $U'$ be the dual space of $U$.

## 2 Decomposition and Variational formulation

In this section, we decompose the system (1.2)-(1.3) into two systems and derive variational forms for solving the decomposed systems.

Let $X$ be the subspace of $\prod_{i=1}^4 H^{\frac{3}{2}}(\Gamma_i)$ defined by the matching condition;

$$h_i(e_{i+1}) = h_{i+1}(e_{i+1}) \ \text{ for } 1 \le i \le 4,$$

where $e_i's$ are the vertices of $\Gamma$ with the convention that $e_5 = e_1$. We shall denote by $\ll \cdot,\cdot \gg$(resp., $< \cdot,\cdot >$) the bilinear form of the duality between $X'$ and $X$(resp., $(\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))'$ and $\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i)$) which is defined by $\ll L, v \gg := L(v),\ L \in X', v \in X$ (resp., $< L, v > := L(v),\ L \in (\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))', v \in \prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i)$). The bilinear form $\ll \cdot,\cdot \gg$ is an extension of $(\cdot,\cdot)_{L^2(\Gamma)}$; $\ll w, v \gg = \int_\Gamma wv\ ds$ for all $v \in X, w \in L^2(\Gamma)$.

Consider the following spaces :

$$
\begin{aligned}
&H(\Omega)=\{u\in L^2(\Omega)\mid \Delta u\in L^2\Omega)\},\\
&\mathcal{H}=\{u\in L^2(\Omega)\mid \Delta u=0\},\\
&L_0^2(\Omega)=\{u\in L^2(\Omega)\mid (u,1)=0\},\\
&V=\{u\in H^2(\Omega)\mid \tfrac{\partial u}{\partial \mathbf{n}}|_\Gamma=0\},\\
&G=\{q\in X'\mid \ll q,1\gg=0\}.
\end{aligned}
$$

We now decompose the system (1.2)-(1.3) into the following two problems. Let $\bar\psi,\bar\omega$ be the solutions of the problem:

$$
\begin{aligned}
&-\nu\Delta\bar\omega=f_1\ \text{ in }\Omega,\\
&-\Delta\bar\psi=\bar\omega\ \text{ in }\Omega,\\
&\bar\psi=0\ \text{ on }\Gamma_i\\
&\tfrac{\partial\bar\psi}{\partial\mathbf{n}_i}=g_i\ \text{ on }\Gamma_i\ \text{ for }1\le i\le 4,
\end{aligned}
\tag{2.1}
$$

and $\tilde\psi,\tilde\omega$ be the solutions of the problem:

$$
\begin{aligned}
&-\nu\Delta\tilde\omega=f_2\ \text{ in }\Omega,\\
&-\Delta\tilde\psi=\tilde\omega\ \text{ in }\Omega,\\
&\tilde\psi=h_i\ \text{ on }\Gamma_i\\
&\tfrac{\partial\tilde\psi}{\partial\mathbf{n}_i}=0\ \text{ on }\Gamma_i\ \text{ for }1\le i\le 4,
\end{aligned}
\tag{2.2}
$$

where curl$\mathbf{f}=f_1+f_2,\ f_1\in L^2(\Omega), f_2\in L_0^2(\Omega)$. Then $\psi=\bar\psi+\tilde\psi$ and $\omega=\bar\omega+\tilde\omega$.

We now derive variational forms for the systems (2.1) and (2.2). Let $(-\Delta_d)^{-1}\in\mathcal{L}\ (L^2(\Omega),H^2(\Omega)\cap H_0^1(\Omega))$ denote Green's operator related to the Dirichlet boundary value problem for $-\Delta$ in $\Omega\subset\mathbf{R}^2$, i.e., for $f\in L^2(\Omega)$, $u=(-\Delta_d)^{-1}f$ is the solution of

$$-\Delta u=f\ \text{ in }\Omega,\ \ u=0\ \text{ on }\Gamma,$$

and $(-\Delta_n)^{-1}\in\mathcal{L}(L_0^2(\Omega),V\cap L_0^2(\Omega))$ denote Green's operator related to the Neumann boundary value problem for $-\Delta$ in $\Omega\subset\mathbf{R}^2$; for $f\in L_0^2(\Omega)$, $u=(-\Delta_n)^{-1}f$ is the solution of

$$-\Delta u=f\ \text{ in }\Omega,\ \ \frac{\partial u}{\partial\mathbf{n}}=0\ \text{ on }\Gamma.$$

Let $\gamma_0, \gamma_1$ be the following trace operators :

$$\gamma_0 v = (v|_{\Gamma_1}, \cdots, v|_{\Gamma_4}),\ \gamma_1 v = (\frac{\partial v}{\partial \mathbf{n}}|_{\Gamma_1}, \cdots, \frac{\partial v}{\partial \mathbf{n}}|_{\Gamma_4}).$$

Assume that curl$\mathbf{f} \in L^2(\Omega)$, $(h_1, \cdots, h_4) \in X$ and $(g_1, \cdots, g_4) \in \prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i)$. Then (1.1) is equivalent to (1.2)–(1.3).

We define an operator $E$ on $L^2(\Omega)$ by

$$E\phi = \gamma_1((-\Delta_d)^{-1}\phi) \ \text{ for } \phi \in L^2(\Omega),$$

and an operator $T$ on $L_0^2(\Omega)$ by

$$T\phi = \gamma_0((-\Delta_n)^{-1}\phi) \ \text{ for } \phi \in L_0^2(\Omega).$$

Then the operator $E$(resp., $T$) is a continuous linear operator from $L^2(\Omega)$ to $\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i)$ (resp., from $L_0^2(\Omega)$ to $X$). Hence, the adjoint operator $E^*$ of $E$ is from $(\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))'$ to $L^2(\Omega)$, and it is given by

$$(E^*\mu, \phi) = < \mu, E\phi > \ \text{ for } \mu \in (\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))',\ \phi \in L^2(\Omega),$$

and the adjoint operator $T^* : X' \to L_0^2(\Omega)'$ of $T$ is given by

$$(T^*\mu)(\phi) = \ll \mu, T\phi \gg \quad \text{for } \mu \in X', \phi \in L_0^2(\Omega).$$

Then we have the following.

(1) For any $\mu \in (\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))'$, let $\tilde{\mu}$ be the unique solution of the problem:

$$\begin{aligned} \Delta\tilde{\mu} &= 0 \quad \text{in } \Omega, \\ \gamma_0\tilde{\mu} &= \mu \quad \text{on } \Gamma, \end{aligned} \tag{2.3}$$

then $E^*\mu = -\tilde{\mu}$.

(2) For any $\eta \in G$, let $\tilde{\eta}$ be the unique solution of the problem:

$$\begin{aligned} \Delta\tilde{\eta} &= 0 \quad \text{in } \Omega, \\ \gamma_1\tilde{\eta} &= \eta \quad \text{on } \Gamma, \\ (\tilde{\eta}, 1) &= 0, \end{aligned} \tag{2.4}$$

then $T^*\eta = \tilde{\eta}$ in the $(L_0^2(\Omega))'$-sense.

By applying $(-\Delta_d)^{-1}\phi$(resp., $(-\Delta_n)^{-1}\phi$), and then Green's second identity and the duality of $E$(resp., $T$), we have the variational form for (2.1)(resp., (2.2)):

$$(\bar{\omega},\phi) = -(E^*q,\phi) + (\frac{1}{\nu}(-\Delta_d)^{-1}f_1,\phi) \text{ for any } \phi \in L^2(\Omega), \tag{2.5}$$

$$(\tilde{\omega},\phi) = (T^*p,\phi) + (\frac{1}{\nu}(-\Delta_n)^{-1}f_2,\phi) \text{ for any } \phi \in L_0^2(\Omega), \tag{2.6}$$

where $q := \gamma_0\bar{\omega} \in (\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))'$ and $p := \gamma_1\tilde{\omega} \in G$. Here, $q$ and $p$ satisfy the following linear variational equations:

$$\begin{aligned}(E^*q,E^*\mu) &= (\tfrac{1}{\nu}(-\Delta_d)^{-1}f_1,E^*\mu) - \mu((g_1,g_2,g_3,g_4)) \\ &\qquad \text{for any } \mu \in (\textstyle\prod_{i=1}^4 H^{\frac{1}{2}}(\Gamma_i))', \\ &\text{where } \mu((g_1,g_2,g_3,g_4)) =< \mu,(g_1,g_2,g_3,g_4) >,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}(T^*p,T^*\eta) &= -(\tfrac{1}{\nu}(-\Delta_n)^{-1}f_2,T^*\eta) + \eta((h_1,h_2,h_3,h_4)) \text{ for any } \mu \in G, \\ &\text{where } \eta((h_1,h_2,h_3,h_4)) =\ll \eta,(h_1,h_2,h_3,h_4) \gg .\end{aligned} \tag{2.8}$$

Therefore, the solution procedure for (2.1)–(2.2) is following:

(1) Compute $\frac{1}{\nu}(-\Delta_d)^{-1}f_1$ and $\frac{1}{\nu}(-\Delta_n)^{-1}f_2$.

(2) Compute $q$, $p$ from (2.7) and (2.8).

(3) Finally, compute $\bar{\omega}$, $\tilde{\omega}$ and $\bar{\psi}$, $\tilde{\psi}$ from (2.5), (2.6) and

$$-\Delta\bar{\psi} = \bar{\omega}, \quad \bar{\psi}\,|_\Gamma = 0, \tag{2.9}$$

$$\tilde{\psi} = (-\Delta_n)^{-1}\tilde{\omega} + \frac{1}{4}\left(\sum_{i=1}^4 \int_{\Gamma_i} h_i\, ds - \ll \gamma_0((-\Delta_n)^{-1}\tilde{\omega}),1 \gg\right). \tag{2.10}$$

Then the problem (2.1)(resp.,(2.2)) is equivalent to (2.5), (2.7) and (2.9) (resp., (2.6), (2.8) and (2.10)). An application of these abstract forms to sine approximation, see [11].

**Remark**. Since $q = \gamma_0 \bar{\omega}$ and $p = \gamma_1 \tilde{\omega}$ are computed from the boundary data $g_i$s and $h_i$s and the actions of $E^*$ and $T^*$ (see equations (2.7) and (2.8)), once $p$ and $q$ are obtained, $\bar{\omega}$, $\bar{\psi}$, $\tilde{\omega}$ and $\tilde{\psi}$ can be computed directly from equations (2.5), (2.6), (2.9) and (2.10). Thus, the main problem to be solved is to compute $q$ and $p$, in other words, the construction $E^*$ and $T^*$ through a Legendre tau approximation.

## 3 Legendre tau approximation scheme

In this section we describe a Legendre tau approximation and factorization scheme. Since the approximation scheme for (2.5), (2.7) and (2.9) can be described in a similar way, we present only the approximation scheme for (2.6), (2.8) and (2.10).

Let $D$ be a subset in $\mathbf{R}$ or $\mathbf{R}^2$. For any nonnegative integer $M$ we denote by $S_M(D)$ the space of all polynomials on $D$ of degree $\leq M$ in each variables. Denoting by $S_M^{1,0}(D)$ the subspace of $S_M(D)$ of all polynomials whose derivatives vanish on $\partial\Omega$. The Legendre polynomial $L_k(x)$, $k \geq 0$, is orthogonal to any Legendre polynomial $L_l, l \neq k$, in $L^2(-1,1)$, it has degree $k$, $L_k(1) = 1$, and satisfies $\int_{-1}^1 L_k^2(x)dx = \frac{2}{2k+1}$ and $\frac{\partial L(\pm 1)}{\partial x} = (\pm 1)^k \frac{k(k+1)}{2}$. Let $P_M$ be the orthogonal projection operator in $L^2(D)$ onto $S_M(D)$. To simplify our expression, we assume that $M$ is even.

We first consider a Legendre tau scheme for the following problem:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \text{ on } \Gamma. \end{aligned} \tag{3.1}$$

Let the tau approximate solution for (3.1) be

$$u_M(x,y) = \sum_{k=0}^{M} \sum_{l=0}^{M} u_{kl} L_k(x) L_l(y).$$

Note that the test functions do not satisfy the boundary conditions individually. Thus, it is necessary to have weighted residual conditions for both the PDE and the boundary conditions. From the weighted residual conditions for the boundary conditions, we have

$$
\begin{aligned}
u_{k,M-1}\frac{(M-1)M}{2} &= -\sum_{l=1,l:\text{odd}}^{M-3} u_{kl}\frac{l(l+1)}{2}, \quad k=0,\cdots,M,\\
u_{kM}\frac{M(M+1)}{2} &= -\sum_{l=2,l:\text{even}}^{M-2} u_{kl}\frac{l(l+1)}{2}, \quad k=0,\cdots,M,\\
u_{M-1,l}\frac{(M-1)M}{2} &= -\sum_{k=1,k:\text{odd}}^{M-3} u_{kl}\frac{k(k+1)}{2}, \quad l=0,\cdots,M,\\
u_{Ml}\frac{M(M+1)}{2} &= -\sum_{k=2,k:\text{even}}^{M-2} u_{kl}\frac{k(k+1)}{2}, \quad l=0,\cdots,M.
\end{aligned} \tag{3.2}
$$

Boundary equations (3.2) give only $4M$ independent equations since the four corners of the square have been counted twice. From (3.2), the coefficients $u_{M-1,l}$, $u_{k,M-1}$, $u_{Ml}$ and $u_{kM}, k,l=0,\cdots,M$ are determined by the coefficients $u_{kl}, k,l=0,\cdots,M-2$, so that we need $(M-1)^2=(M+1)^2-4M$ equations to determine the unknown coefficients $u_{kl}, k,l=0,\cdots,M$, completely. Therefore, the spectral tau approximation of the problem (3.1) is equivalent to

$$
\begin{aligned}
\text{find} \qquad & u_M \in S_M^{1,0}(\Omega)\cap L_0^2(\Omega) \text{ such that}\\
& (-\Delta u_M,\phi)=(f,\phi) \quad \text{for any } \phi\in S_{M-2}(\Omega).
\end{aligned} \tag{3.3}
$$

To introduce our diagonalization technique for the Legendre tau approximation scheme, we represent (3.3) as a linear system. We recall some properties (see [6]) which will be of constant use. The formal expansion of the first derivative of a function $v(x)=\sum_{k=0}^{\infty}\hat{v}_kL_k(x)$ can be written as

$$
\frac{dv(x)}{dx}=\sum_{m=0}^{\infty}\hat{v}_m^1L_m(x), \tag{3.4}
$$

where

$$
\hat{v}_m^1=(2m+1)\sum_{k=m+1 k+m:\text{odd}}^{\infty}\hat{v}_k.
$$

Actually, this formula can be justified for every $v(x)\in H^1(-1,1)$ (see [7]). First, we shall construct the matrix $A$ with size of $M\times(M+1)$ representing the differential operator $\frac{d}{dx}$ on the space $S_M(-1,1)$. Let $v(x):=\sum_{k=0}^{M}v_kL_k(x)$ and $\frac{dv(x)}{dx}:=\sum_{k=0}^{M-1}a_kL_k(x)$.

Then, from relation (3.4), define the matrix $A$ by

(3.5)
$$
\begin{aligned}
A \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ v_{M-1} \\ v_M \end{bmatrix} &:= \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & \cdot & \cdot & 1 & 0 \\ 0 & 0 & 3 & 0 & 3 & \cdot & \cdot & 0 & 3 \\ 0 & 0 & 0 & 5 & 0 & \cdot & \cdot & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 & \cdot & \cdot & 0 & 7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 2M-3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 2M-1 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ v_{M-1} \\ v_M \end{bmatrix} \\
&= \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdot & \cdot & a_{M-2} & a_{M-1} \end{bmatrix}^T .
\end{aligned}
$$

Second, we shall construct the matrix $A_0$ of size $M \times (M-1)$ representing the differential operator $\frac{d}{dx}$ on the space $S_M^{1,0}(-1,1)$. Let $v \in S_M^{1,0}(-1,1)$. Since $\frac{\partial v}{\partial x}(\pm 1) = 0$, we have the following relation.

(3.6)
$$
\begin{aligned}
v_M \tfrac{M(M+1)}{2} &= -(v_2 \tfrac{2\cdot 3}{2} + v_4 \tfrac{4\cdot 5}{2} + \cdots + v_{M-2} \tfrac{(M-2)\cdot(M-1)}{2}), \\
v_{M-1} \tfrac{(M-1)M}{2} &= -(v_1 \tfrac{1\cdot 2}{2} + v_3 \tfrac{3\cdot 4}{2} + v_5 \tfrac{5\cdot 6}{2} + \cdots + v_{M-3} \tfrac{(M-3)\cdot(M-2)}{2}).
\end{aligned}
$$

By using (3.6), define the matrix $A_0$ by

$$
\begin{aligned}
A_0 \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix} := & \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & 0 & 3 & 0 & 3 & 0 & \cdot & \cdot & 3 \\ 0 & 0 & 0 & 5 & 0 & 5 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & \cdot & \cdot & 7 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 2M-5 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix} \\
& - \begin{bmatrix} 1 \\ 0 \\ 5 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 2M-3 \\ 0 \end{bmatrix} \overbrace{\begin{bmatrix} 0 & \frac{1\cdot 2}{2} & 0 & \frac{3\cdot 4}{2} & \cdot & \cdot & \frac{(M-3)\cdot(M-2)}{2} & 0 \end{bmatrix}}^{M-1} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix} \\
& - \begin{bmatrix} 0 \\ 3 \\ 0 \\ 7 \\ \cdot \\ \cdot \\ 2M-5 \\ 0 \\ 2M-1 \end{bmatrix} \overbrace{\begin{bmatrix} 0 & 0 & \frac{2\cdot 3}{2} & 0 & \frac{4\cdot 5}{2} & \cdot & \cdot & \frac{(M-2)\cdot(M-1)}{2} \end{bmatrix}}^{M-1} \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix} .
\end{aligned}
\tag{3.7}
$$

Third, we shall construct the matrix $B$ representing the differential operator $-\frac{d^2}{dx^2}$ on the space $S_M^{1,0}(-1,1)$. The matrix $B$ must have size of $(M-1)\times(M-1)$. Let $v(x) \in S_M^{1,0}(-1,1)$ and $\phi(x) := \sum_{k=0}^{M-2} \phi_k L_k(x) \in S_{M-2}(-1,1)$. Then there exists a

$\psi(x) := \phi(x) + \alpha L_{M-1}(x) + \beta L_M(x) \in S_M^{1,0}(-1,1)$ so that

$$\int_{-1}^{1} -\frac{d^2v(x)}{dx^2}\phi(x)dx = \int_{-1}^{1} -\frac{d^2v(x)}{dx^2}\psi(x)dx = \int_{-1}^{1} \frac{dv(x)}{dx}\frac{d\psi(x)}{dx}dx. \tag{3.8}$$

Let $\psi(x) := \sum_{k=0}^{M} \psi_k L_k(x)$. Then $\psi_k = \phi_k, 0 \le k \le M-2$, and $\psi_{M-1} = \alpha, \psi_M = \beta$. If we let

$$\begin{aligned} &\tfrac{d\psi(x)}{dx} := \sum_{k=0}^{M-1} b_k L_k(x), \\ &[\tilde{v}] := [v_0 \ \ v_1 \ \ \cdots \ \ v_{M-2}]^T, \\ &[\tilde{\psi}] := [\psi_0 \ \ \psi_1 \ \ \cdots \ \ \psi_{M-2}]^T, \end{aligned}$$

then from (3.7), we have

$$A_0[\tilde{v}] = [a_0 \ \ a_1 \ \ \cdots \ a_{M-1}]^T, \ \ A_0[\tilde{\psi}] = [b_0 \ \ b_1 \ \cdots \ b_{M-1}]^T. \tag{3.9}$$

If we let

$$-\frac{d^2v(x)}{dx^2} := \sum_{k=0}^{M-2} \beta_k L_k(x),$$

then by orthogonality, we have

$$\begin{aligned} &\int_{-1}^{1} -\frac{d^2v(x)}{dx^2}\phi(x)dx = \sum_{k=0}^{M-2} \beta_k\phi_k\frac{2}{2k+1}, \\ &\int_{-1}^{1} \frac{dv(x)}{dx}\frac{d\psi(x)}{dx}dx = \sum_{k=0}^{M-1} a_k b_k \frac{2}{2k+1}. \end{aligned} \tag{3.10}$$

We define the matrices $Q$ and $\bar{Q}$ by

$$Q := \begin{bmatrix} 2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \frac{2}{2M-1} \end{bmatrix}_{M\times M}, \qquad \bar{Q} := \begin{bmatrix} 2 & 0 & 0 & 0 & \cdot & \cdot & 0 \\ 0 & \frac{2}{3} & 0 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & \frac{2}{5} & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \frac{2}{7} & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \frac{2}{2M-3} \end{bmatrix}_{(M-1)\times(M-1)}.$$

From (3.8), (3.9) and (3.10), we have

$$\begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix}^T A_0^T Q A_0 \begin{bmatrix} \psi_0 \\ \psi_1 \\ \cdot \\ \cdot \\ \psi_{M-2} \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \cdot \\ \cdot \\ \beta_{M-2} \end{bmatrix}^T \bar{Q} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \phi_{M-2} \end{bmatrix}.$$

Since

$$B[\tilde{v}] = [\beta_0 \;\; \beta_1 \;\; \cdots \;\; \beta_{M-2}]^T,$$

we have

$$\begin{aligned} \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix}^T A_0^T Q A_0 \begin{bmatrix} \psi_0 \\ \psi_1 \\ \cdot \\ \cdot \\ \psi_{M-2} \end{bmatrix} &= \left[ [B] \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix} \right]^T \bar{Q} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \phi_{M-2} \end{bmatrix} \\ &= \begin{bmatrix} v_0 \\ v_1 \\ \cdot \\ \cdot \\ v_{M-2} \end{bmatrix}^T [B]^T \bar{Q} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \cdot \\ \cdot \\ \phi_{M-2} \end{bmatrix}. \end{aligned}$$

Since $v(x)$ and $\phi(x)$ are arbitrary, and $\psi_k = \phi_k, 0 \le k \le M-2$, we obtain

$$B^T \bar{Q} = A_0^T Q A_0,$$

so that

$$B = \bar{Q}^{-1} A_0^T Q A_0. \tag{3.11}$$

Now we are ready to describe the problem (3.3) into a linear system. Let $u_M(x,y) := \sum_{k=0}^{M} \sum_{l=0}^{M} u_{kl} L_k(x) L_l(y)$ with $u_{00} = 0$, be the tau solution of the problem (3.3). Let $\tilde{U}$ and $U$ be defined by

$$U = \begin{bmatrix} u_{00} & \cdots & u_{0M} \\ \vdots & \vdots & \vdots \\ u_{M0} & \cdots & u_{MM} \end{bmatrix}, \; \tilde{U} = \begin{bmatrix} u_{00} & \cdots & u_{0,M-2} \\ \vdots & \vdots & \vdots \\ u_{M-2,0} & \cdots & u_{M-2,M-2} \end{bmatrix}.$$

Hence, from the orthogonal property of Legendre polynomials, we have

$$P_{M-2}(-\Delta u_M(x,y)) = \sum_{k=0}^{M-2}\sum_{l=0}^{M-2} \alpha_{kl} L_k(x) L_l(y), \tag{3.12}$$

where

$$\begin{bmatrix} \alpha_{00} & \cdots & \alpha_{0,M-2} \\ \vdots & \vdots & \vdots \\ \alpha_{M-2,0} & \cdots & \alpha_{M-2,M-2} \end{bmatrix} = B\tilde{U} + \tilde{U}B^T.$$

Let $P_{M-2}f(x,y) := \sum_{k=0}^{M-2}\sum_{l=0}^{M-2} f_{kl}L_k(x)L_l(y)$ and let $f_{00}=0$. From (3.3) and (3.12), we have a spectral tau solver for the two-dimensional Poisson equation:

$$B\tilde{U} + \tilde{U}B^T = F, \tag{3.13}$$

where

$$F := [f_{kl}]_{(M-1)\times(M-1)}.$$

We now present our diagonalization technique based on Schur decomposition for the linear system (3.13). The successful implementation requires the previous procedures to keep up the merits and to avoid the faults of a matrix diagonalization and Schur decomposition. Let $H = A_0^T Q A_0$, so that $H$ is a symmetric matrix. Then (3.13) can be expressed as

$$[\bar{Q}^{-1}H]\tilde{U} + \tilde{U}[\bar{Q}^{-1}H]^T = F. \tag{3.14}$$

Let

$$\tilde{H} := \bar{Q}^{-\frac{1}{2}} H \bar{Q}^{-\frac{1}{2}},\ \tilde{\tilde{U}} := \bar{Q}^{\frac{1}{2}}\tilde{U}\bar{Q}^{\frac{1}{2}},\ \tilde{F} := \bar{Q}^{\frac{1}{2}} F \bar{Q}^{\frac{1}{2}},$$

and multiply both sides of (3.14) by $\bar{Q}^{\frac{1}{2}}$, then we have

$$\tilde{H}\tilde{\tilde{U}} + \tilde{\tilde{U}}\tilde{H} = \tilde{F}. \tag{3.15}$$

By construction of $\tilde{H}$, there exist an orthogonal matrix $V$ and a diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_{M-1}),\ \lambda_1 = 0, \lambda_i > 0, i = 2, \cdots, M-1$, such that $\tilde{H}V = VD$. If we let

$$\tilde{\tilde{\tilde{U}}} := V^T\tilde{\tilde{U}}V,\ \tilde{\tilde{F}} := V^T\tilde{F}V,$$

(3.15) becomes

$$D\tilde{\tilde{\tilde{U}}} + \tilde{\tilde{\tilde{U}}}D = \tilde{\tilde{F}}. \tag{3.16}$$

Let $\tilde{\tilde{F}} := [\tilde{\tilde{f}}_{kl}]_{(M-1)\times(M-1)}$ and $\tilde{\tilde{\tilde{U}}} = [\tilde{\tilde{\tilde{u}}}_{kl}]$, from (3.16), we have the following relation :

$$\begin{cases} \tilde{\tilde{\tilde{u}}}_{kl} = 0 \ \text{ if } k = l = 0, \\ \tilde{\tilde{\tilde{u}}}_{kl} = \frac{\tilde{\tilde{f}}_{kl}}{\lambda_k + \lambda_l} \ \text{ otherwise.} \end{cases}$$

Since $\tilde{\tilde{U}} = V\tilde{\tilde{\tilde{U}}}V^T$ and $\tilde{U} = \bar{Q}^{-\frac{1}{2}}\tilde{\tilde{U}}\bar{Q}^{-\frac{1}{2}}$, the coefficients $u_{kl}, 0 \le k, l \le M-2$, are obtained through the following matrix multiplication only without solving the linear system (3.13) or (3.15).

$$\begin{aligned} \tilde{U} &= \bar{Q}^{-\frac{1}{2}}V\tilde{\tilde{\tilde{U}}}V^T\bar{Q}^{-\frac{1}{2}} \\ &= [\bar{Q}^{-\frac{1}{2}}V]\tilde{\tilde{\tilde{U}}}[\bar{Q}^{-\frac{1}{2}}V]^T. \end{aligned} \tag{3.17}$$

Let $C$ be the matrix representing the discrete operator $T_M^*$ of $T^*$. To obtain the matrix $C$, we extend the tau solver based on our diagonalization technique for the homogeneous case to the case with inhomogeneous boundary condition. The matrix $C$ consists of four column blocks. To obtain, for example, the $n$th column of the first block, we consider the following problem:

$$\begin{cases} \Delta u = 0 \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \mathbf{n}} = L_n(x)\chi_{\Gamma_1} \ \text{ on } \Gamma. \end{cases} \tag{3.18}$$

Let $u_M := \sum_{k=0}^{M}\sum_{l=0}^{M} u_{kl}L_k(x)L_l(y)$ be the approximate solution for the problem (3.18). Since the test functions do not satisfy the boundary conditions individually, the weighted residual conditions corresponding to the boundary conditions are

$$
\begin{aligned}
u_{k,M-1}\tfrac{M(M-1)}{2} &= -\sum_{l=1,l:\mathrm{odd}}^{M-3} u_{kl}\frac{l(l+1)}{2} \ \text{ for } k=0,\cdots,M,\ k\neq n,\\
u_{kM}\tfrac{M(M+1)}{2} &= -\sum_{l=2,l:\mathrm{even}}^{M-2} u_{kl}\frac{l(l+1)}{2} \ \text{ for } k=0,\cdots,M,\ k\neq n,\\
u_{M-1,l}\tfrac{M(M-1)}{2} &= -\sum_{k=1,k:\mathrm{odd}}^{M-3} u_{kl}\frac{k(k+1)}{2} \ \text{ for } l=0,\cdots,M,\\
u_{Ml}\tfrac{M(M+1)}{2} &= -\sum_{k=2,k:\mathrm{even}}^{M-2} u_{kl}\frac{k(k+1)}{2} \ \text{ for } l=0,\cdots,M,\\
u_{n,M-1}\tfrac{M(M-1)}{2} &= \tfrac{1}{2}-\sum_{l=1,l:\mathrm{odd}}^{N-3} u_{nl}\frac{l(l+1)}{2},\\
u_{nM}\tfrac{M(M+1)}{2} &= -\tfrac{1}{2}-\sum_{l=2,l:\mathrm{even}}^{N-2} u_{nl}\frac{l(l+1)}{2}.
\end{aligned} \tag{3.19}
$$

Let

$$u_M^2 = \frac{1}{M(M-1)}L_n(x)L_{M-1}(y) - \frac{1}{M(M-1)}L_n(x)L_M(y) \in L_0^2(\Omega),$$

and $u_M^1$ be the Legendre tau approximate solution of the following problem:

$$\begin{cases} -\Delta u^1 = \Delta u_M^2 \ \text{ in } \Omega, \\ \frac{\partial u^1}{\partial \mathbf{n}} = 0 \ \text{ on } \Gamma. \end{cases} \tag{3.20}$$

Then $u_M^1 \in S_M^{1,0}(\Omega) \cap L_0^2(\Omega)$ satisfies

$$(-\Delta u_M^1, \phi) = (\Delta u_M^2, \phi) \quad \text{for } \phi \in S_{M-2}(\Omega), \tag{3.21}$$

and for $n = M-1, M$,

$$(\Delta u_M^2, \phi) = 0 \quad \text{for } \phi \in S_{M-2}(\Omega). \tag{3.22}$$

From (3.22), we have a trivial solution $u_M^1 = 0$ for $n = M-1, M$. To avoid the trivial solution, for $N \le M-2$, we define the boundary element space $\mathcal{B}_N \subset L_0^2(\Gamma)$ on $\Gamma$ by

$$\mathcal{B}_N := \{L_k(x)\chi_{\Gamma_i}, L_l(y)\chi_{\Gamma_j} \mid k,l = 1,\cdots,N, i=1,3, j=2,4\}.$$

Let $P_N^{\mathcal{B}}$ be the $L^2$-projection operator from $L^2(\Gamma)$ to $\mathcal{B}_N$. From the definition of the boundary element space $\mathcal{B}_N$,

$$P_N^{\mathcal{B}}\gamma_1(u_M^1 + u_M^2) = P_N^{\mathcal{B}}\gamma_1(u_M^2) = L_n(x)\chi_{\Gamma_1}. \tag{3.23}$$

From (3.21) and (3.23), $u_M^1 + u_M^2$ becomes the spectral Legendre tau approximate solution of the problem (3.18); $u_M = u_M^1 + u_M^2$. Similarly, we can construct the other columns and those of other blocks of the matrix $C$.

Now we are ready to describe the discretized formulations for (2.6), (2.8) and (2.10) by using the Laplace solver for the homogeneous and inhomogeneous boundary conditions. Define the discrete operator $(-\Delta_n)_M^{-1}$ of $(-\Delta_n)^{-1}$ by

$$(-\Delta_n)_M^{-1}(f) := u_M \quad \text{for } f \in L_0^2(\Omega),$$

where $u_M \in S_M^{1,0}(\Omega) \cap L_0^2(\Omega)$ satisfies:

$$(-\Delta u_M, \phi) = (f, \phi) \quad \text{for } \phi \in S_{M-2}(\Omega),$$

and define the discrete operator $T_M^*$ of $T^*$ by

$$T_M^* \lambda_N := (-\Delta_n)_M^{-1}(\Delta u_2) + u_2 \quad \text{for } \lambda_N \in \mathcal{B}_N,$$

where $u_1 \in S_M^{1,0}(\Omega) \cap L_0^2(\Omega)$, $u_2 \in S_M(\Omega) \cap L_0^2(\Omega)$ satisfy

$$\begin{cases} (-\Delta u_1, \phi) = (\Delta u_2, \phi) \quad \text{for } \phi \in S_{M-2}(\Omega), \\ P_N^{\mathcal{B}} \gamma_1(u_2) = \lambda_N. \end{cases}$$

Therefore, we consider the following discretized formulations for the discrete normal derivative $p_N$ of the vorticity, the discrete vorticity $\omega_M$ and stream function $\psi_M$:

$$\begin{cases} \text{find} & p_N \in \mathcal{B}_N \text{ such that} \\ & (T_M^* p_N, T_M^* \lambda_N) = -(\frac{1}{\nu}(-\Delta_n)_M^{-1}(f_2), T_M^* \lambda_N) + \sum_{i=1}^4 \int_{\Gamma_i} h_i \lambda_N \, ds \\ & \qquad\qquad \text{for any } \lambda_N \in \mathcal{B}_N, \\ \text{find} & \omega_M \in S_{M-2} \text{ such that} \\ & (\omega_M, \phi) = (T_M^* p_N, \phi) + (\frac{1}{\nu}(-\Delta_n)_M^{-1}(f_2), \phi) \quad \text{for any } \phi \in S_{M-2}(\Omega), \\ \text{find} & \psi_M \in S_M(\Omega) \text{ such that} \\ & \psi_M = (-\Delta_n)_M^{-1}(\omega_M) + \frac{1}{4}(\sum_{i=1}^4 \int_{\Gamma_i} h_i \, ds - \int_\Gamma (-\Delta_n)_M^{-1}(\omega_M) \, ds). \end{cases}$$

# 4 Numerical results

The first example is the case with $u = v = 0$ on the sides $\Gamma_1, \Gamma_2, \Gamma_4$, and $u = -(x+1)^2(x-1)^2, v = 0$ on $\Gamma_3$ (see Figure 1 for $\Gamma_i$). This problem has a regular solution. Table

1 shows the convergence for the discrete boundary value $q_N$, the discrete vorticity $\omega_M$ and the discrete stream function $\psi_M$ for curl$\mathbf{f} = 0$. In Figures 2.1 and 2.2, streamlines and vector fields for this case are shown for $M = 16, N = 12$.

Table 1. Convergence of $\omega_M$, $\psi_M$ and $q_N$: curl$\mathbf{f} = 0$, $u = v = 0$ on $\Gamma_1, \Gamma_2, \Gamma_4$, $u = -(x+1)^2(x-1)^2, v = 0$ on $\Gamma_3$, and $M_i = N_i + 4, i = 1, 2$.

| | $\|\omega_{M_1} - \omega_{M_2}\|_{L^2(\Omega)}$ | $\|\psi_{M_1} - \psi_{M_2}\|_{L^2(\Omega)}$ | $\|q_{N_1} - q_{N_2}\|_{L^2(\Gamma)}$ |
|---|---|---|---|
| $M_1 = 20 \quad M_2 = 24$ | 0.1061e-3 | 0.3307e-7 | 0.4807e-3 |
| $M_1 = 24 \quad M_2 = 28$ | 0.0436e-3 | 0.0636e-7 | 0.1488e-3 |
| $M_1 = 28 \quad M_2 = 32$ | 0.0208e-3 | 0.0265e-7 | 0.1114e-3 |
| $M_1 = 32 \quad M_2 = 36$ | 0.0110e-3 | 0.0113e-7 | 0.0739e-3 |
| $M_1 = 36 \quad M_2 = 40$ | 0.0049e-3 | 0.0047e-7 | 0.0449e-3 |

The second example is the case with $\psi = 1$, $\frac{\partial \psi}{\partial \mathbf{n}} = 0$ on $\Gamma$ and curl$\mathbf{f} = L_2(x)L_2(y)$, where $L_2(x)$ and $L_2(y)$ are the second order Legendre polynomials. Table 2 shows the convergence for the discrete normal derivative $p_N$, the discrete vorticity $\omega_M$ and the discrete stream function $\psi_M$. In Figures 3.1 and 3.2, streamlines and vector fields for the second example are shown for $M = 16, N = 12$.

Table 2. Convergence of $\omega_M$, $\psi_M$ and $p_N$: curl$\mathbf{f} = L_2(x)L_2(y)$, $\psi = 1$, $\frac{\partial \psi}{\partial \mathbf{n}} = 0$ on $\Gamma$ and $M_i = N_i + 4, i = 1, 2$.

| | $\|\omega_{M_1} - \omega_{M_2}\|_{L^2(\Omega)}$ | $\|\psi_{M_1} - \psi_{M_2}\|_{L^2(\Omega)}$ | $\|p_{N_1} - p_{N_2}\|_{L^2(\Gamma)}$ |
|---|---|---|---|
| $M_1 = 20 \quad M_2 = 24$ | 0.1885e-5 | 0.8187e-6 | 0.9586e-3 |
| $M_1 = 24 \quad M_2 = 28$ | 0.0625e-5 | 0.3614e-6 | 0.6143e-3 |
| $M_1 = 28 \quad M_2 = 32$ | 0.0292e-5 | 0.1668e-6 | 0.3665e-3 |
| $M_1 = 32 \quad M_2 = 36$ | 0.0190e-5 | 0.0842e-6 | 0.2231e-3 |
| $M_1 = 36 \quad M_2 = 40$ | 0.0131e-5 | 0.0458e-6 | 0.1460e-3 |

The third example is the case with $u = -1$ on $\Gamma_3$. Other boundary conditions are the same as in the first example. We do not expect that $\psi$ belongs to $H^2(\Omega)$, because of singularities at the two top corners. However, for the discretized Stokes and Navier-Stokes problems, we expect a $H^2(\Omega)$ solution due to implementation of boundary conditions. This means that $u$ drops to zero continuously along the vertical sides $x = -1, x = 1$ from $y = 1$ to $y = 1 - \varepsilon$ for a small $\varepsilon > 0$. Here, we chose $\varepsilon = 0.15$ for this example. Physically, this smoothing indicates that a small amount of fluid enters the cavity through the point $(1, 1)$ and the same amount of fluid leaves the cavity through the point $(-1, 1)$. This guarantees that the compatibility condition $\int_\Gamma \mathbf{u} \cdot \mathbf{n} ds = 0$ is satisfied. In Figures 4.1 and 4.2, streamlines and vector fields for the third example are shown for $M = 16, N = 12$.

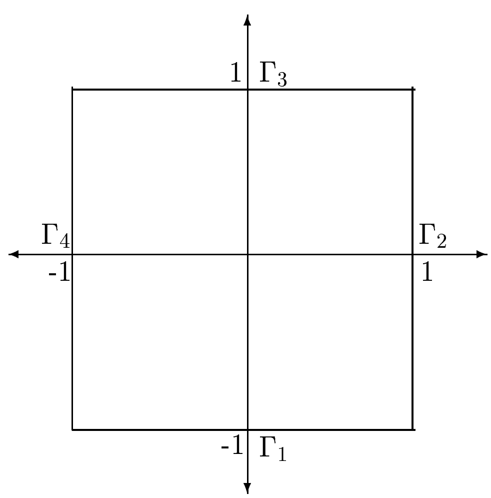


Figure 1. Geometry of the driven cavity problem.

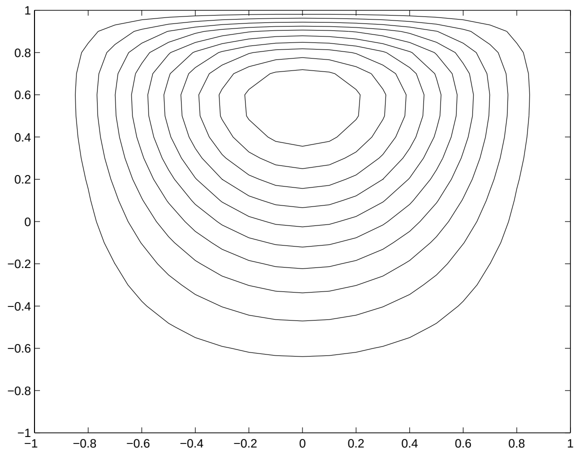


Figure 2.1. Contour of the stream lines (case 1).

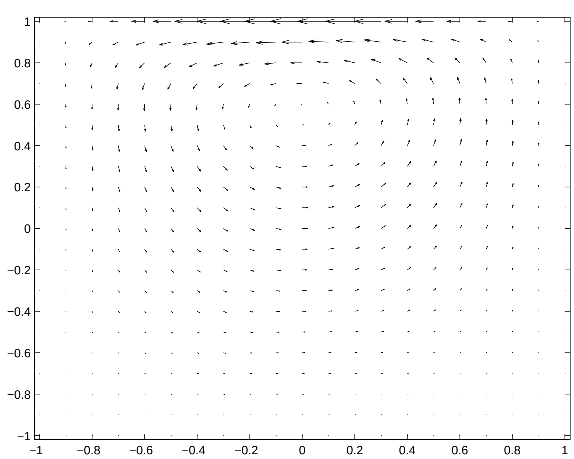


Figure 2.2. Velocity vector fields (case 1).

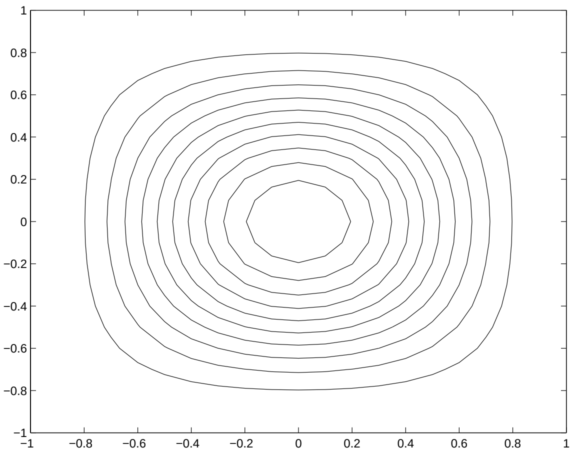


Figure 3.1. Contour of the stream lines (case 2).

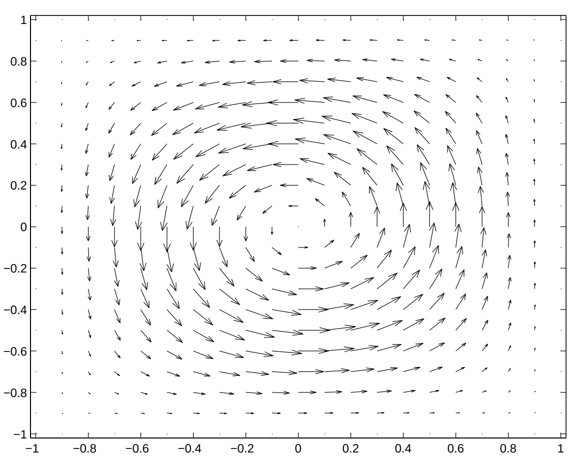


Figure 3.2. Velocity vector fields (case 2).

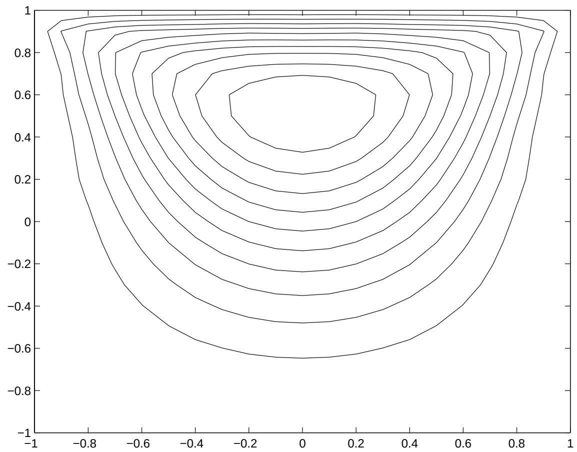


Figure 4.1. Contour of the stream lines (case 3).

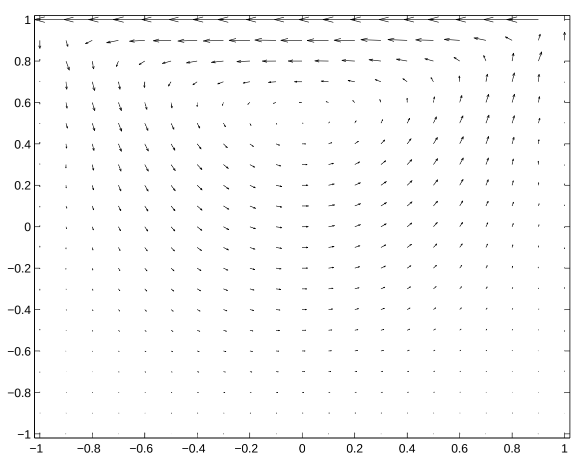


Figure 4.2. Velocity vector fields (case 3).

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