J. KSIAM **Vol.4**, No.2, 85-97, 2000

ON THE CONVERGENCE OF QUADRATURE RULE FOR SINGULAR INTEGRAL EQUATIONS

SEKI KIM

ABSTRACT. A quadrature rule for the solution of Cauchy singular integral equation is constructed and investigated. This method to calculate numerically singular integrals uses classical Jacobi quadratures adopting Hunter's method. The proposed method is convergent under a reasonable assumption on the smoothness of the solution.

1. INTRODUCTION

The singular integral equation with Cauchy kernel most often considered has the form

$$a(x)\ \phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\phi(t)dt}{t-x} + \int_{-1}^{1}k(x,t)\phi(t)dt = f(x) \qquad -1 < x < 1. \tag{1}$$

The first integral term is understood to be a Cauchy principal value integral. It is possible to reduce singular integral equations (hereafter SIE's) to Fredholm integral equations (indirect method), but direct solution methods are preferred in practice. Also it is proved that when the Gauss numerical integration rule is used that both numerical methods are equivalent in the sense that they provide the same numerical results for the same number of abscissae used in numerical integrations [15].

Usually the unknown function is replaced by the product of a smooth function times a function taken as the weight of the quadrature. For variable coefficients SIE's, this is non classical and the nodes and weights of the quadrature rule must be constructed from scratch. But for constant coefficients SIE's this reduces to Jacobi quadrature. In this paper, we want to analyze the replacement of the possibly nonclassical weights and nodes, by the weights and zeros of Jacobi polynomials. This is a quite simpler approach than methods using nonclassical weights and nodes.

We mention some methods for the variable coefficient SIE's. Theocaris and Tsamasphyros [14] attempt to apply a Gauss-Jacobi quadrature rule directly, but this results in the need to compute the zeros of a second kind of Jacobi function. Dow and Elliott [4] have developed an algorithm with error analysis, for solving an approximate solution to (1) by replacing f and k by polynomial approximations. In [12] the solvability of the discrete system is proved for arbitrary selection of quadrature and collocation nodes, but no error analysis is given there. Here we propose a simpler method and consider nodes which are well known zeros of Jacobi polynomial and want to develop the error analysis for the proposed method. This study concerns only global polynomial approximation.

In the error analysis, a restrictive assumption is used, which is a bound on the size of the coefficients. Since we are unable to find a closed form inverse from the matrix of the discretized system, we perform the error analysis by treating the singular operator as a perturbation of the identity.

2. Preliminaries

The second kind singular integral equation with variable coefficients can be written as

$$a(x)\ \phi(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{\mathcal{K}(x,t)}{t-x}\phi(t)dt = f(x) \qquad -1 < x < 1$$

This equation is reduced to (1) by setting $\mathcal{K}(x,t) = (\mathcal{K}(x,t) - \mathcal{K}(x,x)) \ + \ \mathcal{K}(x,x)$ and

$$k(x,t) = \begin{cases} (\mathcal{K}(x,t) - \mathcal{K}(x,x))/(t-x) & t \neq x \\ \mathcal{K}'(x,t) & t = x \end{cases}$$

The singular integral in (1) is interpreted in the Cauchy principal value sense. And the equation

$$a(x)\phi(x) + \frac{b(x)}{\pi}\int_{-1}^{1} \frac{\phi(t)}{t-x}dt = f(x) \quad -1 < x < 1 \tag{2}$$

is called the dominant equation of the equation (1) [10]. We have solutions for SIE's under the following assumptions in general.

- The functions a, b, f and k are Hölder continuous in each independent variable on $[-1,1]$.
- The functions $S(x) = a(x) + b(x)$ and $D(x) = a(x) - b(x)$ do not vanish anywhere on $[-1,1]$.

Also it is not restrictive to assume the coefficients to satisfy $r(x)^2 = a(x)^2 + b(x)^2 = 1$ and $b(x) \neq 0$ on $(-1,1)$. For the latter case, $b(x)$ may vanish at a finite number of isolated points in $(-1,1)$ as long as it remains of one sign. However, we will assume $b(x)$ not vanishing in $(-1,1)$ for simplicity. Following [4], let us define the continuous function

$$\begin{aligned} \theta(x) &= -\frac{1}{2\pi i}\ln\frac{a(t) - ib(t)}{a(t) + ib(t)} \\ &= \frac{1}{\pi}\arctan\frac{b(t)}{a(t)} + N(t) \end{aligned}$$

where N takes only integer values and may have discontinuities at the zeros of a/b and $-\pi/2 < \arctan x < \pi/2$. In order to apply Elliott-Parget quadrature [2, 3], we assume $a(x)$ doesn't have zeros in $(-1,1)$, or we can choose Hunter's quadrature[8, 1]. The fundamental function Z is defined as

$$Z(t) = (1+t)^{n_1}(1-t)^{n_2}\exp\left(-\int_{-1}^{1}\frac{\theta(\tau)}{\tau - t}d\tau\right) \text{ for } t \in (-1,1)$$

where n_1 and n_2 are integers. This function can be rewritten, after fixing n_1 and n_2, as

$$Z(t) = (1+t)^{n_1-\theta(1)}(1-t)^{n_2+\theta(-1)}\ \Omega(t)$$

say, where

$$\Omega(t) = \exp\{(\theta(1)-\theta(t))\ln(1-t)+(\theta(t)-\theta(-1))\ln(1+t) - \int_{-1}^{1}\frac{\theta(\tau)-\theta(t)}{\tau-t}d\tau\}.$$

Here define $\alpha = n_2 - \theta(1)$ and $\beta = n_1 + \theta(-1)$. The behavior of Z near the end points -1 and 1 can be considered as following :
Z is bounded and $1/Z$ is infinite, but integrable if $0 < \alpha, \beta < 1$
Z is infinite, but integrable and $1/Z$ is bounded if $-1 < \alpha, \beta < 0$
In all cases we define the index κ of the singular operator of the dominant equation by

$$\kappa = -(n_1 + n_2).$$

It turns out that κ can have up to three values depending upon whether Z is chosen to be bounded or unbounded at non-special ends i.e. those for which $\theta(1)$ (or $\theta(-1)$) is not an integer. The largest of these values of κ is the index of singular operator [5]. It is shown that this index can attain only three values $-1, 0$ and 1 if $b(x) \neq 0$ in $[-1, 1]$. If $\kappa = 1$, we need an extra condition, to get a unique solution,

$$\frac{1}{\pi}\int_{-1}^{1}\phi(t)dt = C_o. \tag{3}$$

When $\kappa = -1$, the consistency condition for the existence of solution of (2) is

$$\int_{-1}^{1}\frac{f(x)}{Z(x)}dx = 0.$$

We use $\rho(x) = (1-x)^{\alpha}(1+x)^{\beta}$ as a Jacobi weight function. Then $Z(x) = \rho(x)\Omega(x)$ where $\Omega(x)$ is a positive continuous function on $[-1, 1]$. $\phi(x)$ can be rewritten in terms of $\rho(x)$, which expresses explicitly the singular behavior at the end points, and a new unknown function $y^*(x)$ such that

$$\phi(x) = Z(x)\varphi(x) = \rho(x)\Omega(x)\varphi(x) = \rho(x)y^*(x).$$

We consider here the case $\kappa = 1$ i.e $-1 < \alpha, \beta < 0$. Let $H_\mu[-1, 1]$ denote the class of Hölder continuous function of order μ on $[-1, 1]$. Then clearly $\rho(x) \in H_\mu[-1, 1]$ with $\mu = \min(-\alpha, -\beta)$ [10]. The integral (2) can be discretized by a classical Gaussian quadrature. Let t_i be the zeros of $P_n^{(\alpha,\beta)}(x)$ and s_j be the zeros of $P_{n-1}^{(-\alpha,-\beta)}(x)$ where $P_n^{(\alpha,\beta)}(x)$ denotes the Jacobi polynomial of degree n relative to the weight function $\rho(x)$, and $P_{n-1}^{(-\alpha,-\beta)}(x)$ the one relative to $1/\rho(x)$. Since $b(x) \neq 0$, we can rewrite the equation (2) as

$$\pi\frac{a(x)}{b(x)}\rho(x)y^*(x) + \int_{-1}^{1}\frac{\rho(t)y^*(t)}{t-x}dt = \frac{\pi f(x)}{b(x)} \tag{4}$$

If we use $a_*(x)$ for $\pi a(x)/b(x)$ and $f_*(x)$ for $\pi f(x)/b(x)$, then (4) becomes

$$a_*(x)\rho(x)y^*(x) + \int_{-1}^{1}\frac{\rho(t)y^*(t)}{t-x}dt = f_*(x) \tag{5}$$

To evaluate the singular integral in (5), we will use two kinds of Gauss-Jacobi quadrature : Hunter's and Elliott-Paget's, where both methods use Lagrange interpolations to derive quadrature rules. Let

$$\psi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1} \rho(t) \frac{P_n^{(\alpha,\beta)}(t)}{t-z} dt = 2(z-1)^{\alpha}(z+1)^{\beta} q_n^{(\alpha,\beta)}(z) \text{ for } z \notin [-1,1]$$

where $q_n^{(\alpha,\beta)}$ represents the so called Jacobi function of the second kind. We can define the values of the function $\psi_n^{(\alpha,\beta)}(x)$ on the interval $[-1,1]$ as follows

$$\psi_n^{(\alpha,\beta)}(x) = \frac{1}{2}\left\{\psi_n^{(\alpha,\beta)}(x+i0) + \psi_n^{(\alpha,\beta)}(x-i0)\right\}.$$

It can be expressed explicitly by means of the hypergeometric function [13]. Then Hunter's method has the form;

$$Q_n^*(y^*,x) = \sum_{i=1}^{n} \frac{w_i y^*(t_i)}{t_i - x} + \frac{\psi_n^{(\alpha,\beta)}(x) y^*(x)}{P_n^{(\alpha,\beta)}(x)}$$

where

$$\int_{-1}^{1} \frac{\rho(t) y^*(t)}{t-x} dt = Q_n^*(y^*,x) + \epsilon_{G_h} \quad \text{for } x \in [-1,1].$$

The Elliott-Paget method for singular integral is of the form

$$Q_n(y^*,x) = \sum_{i=1}^{n} \left[w_i - \frac{\psi_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}(t_i)} \right] \frac{y^*(t_i)}{t_i - x}$$

where

$$\int_{-1}^{1} \frac{\rho(t) y^*(t)}{t-x} dt = Q_n(y^*,x) + \epsilon_{G_e} \quad \text{for } x \in [-1,1].$$

Let us remark that through the paper, C_i, $i = 0,1,2,\ldots,9$ represent different positive constants.

3. Numerical Scheme

In this section, by applying Hunter's method to (5) and (3) and collocating at the node points s_j, we have

$$\left[a_*(s_j)\rho(s_j) + \frac{\psi_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)}(s_j)} \right] y^*(s_j) + \sum_{i=1}^{n} \frac{w_i y^*(t_i)}{t_i - s_j} + \epsilon_{G_1} = f_*(s_j), \quad j = 1,\ldots,n-1 \tag{6}$$

and

$$\sum_{i=1}^{n} w_i y^*(t_i) = C_o \tag{7}$$

where (i) ϵ_{G_1} is the error of Gauss-Jacobi quadrature on s_j and

$$\text{(ii) } w_i = \int_{-1}^{1} \rho(t) \frac{P_n^{(\alpha,\beta)}(t)}{(t-t_i)P_n^{(\alpha,\beta)'}(t_i)} dt \tag{8}$$

are the weights of Gauss-Jacobi quadrature. (7) is from the normalization condition (3). Recall that this equation is of index 1. We transform (5) into the following equation

$$a_*(x)\rho(x)y^*(x) + \int_{-1}^{1} \frac{1}{\rho(t)} \frac{y^*(t)\rho^2(t)}{t-x} dt = f_*(x)$$

and then proceed similarly, using as weight $1/\rho(t)$ and collocating at t_i,

$$\left[a_*(t_i)\rho(t_i) + \frac{\psi_{n-1}^{(-\alpha,-\beta)}(t_i)}{P_{n-1}^{(-\alpha,-\beta)}(t_i)}\rho^2(t_i)\right] y^*(t_i) + \sum_{j=1}^{n-1} \frac{w_j^* y^*(s_j)}{s_j - t_i}\rho^2(s_j) + \epsilon_{G_2} \tag{9}$$
$$= \quad f_*(t_i) \qquad i = 1, 2, \ldots, n$$

where (i) ϵ_{G_2} is the error of Gauss-Jacobi quadrature on t_i and

$$\text{(ii) } w_j^* = \int_{-1}^{1} \frac{1}{\rho(t)} \frac{P_{n-1}^{(-\alpha,-\beta)}(t)}{(t-s_j)P_n^{(-\alpha,-\beta)'}(s_j)} dt \qquad j = 1, \ldots, n-1$$

are the weights of Gauss-Jacobi quadrature.

In (9), we applied a different weight function $1/\rho(x)$ to the exactly same equation (5) and we are looking for the solution in $(-1, 1)$ because we already knew the end point behavior of the solution. Note that $\rho(x)$ is smooth on $(-1, 1)$ i.e. $y^*(x)\rho^2(x)$ can be assumed to be smooth on $(-1, 1)$ if $y^*(x)$ is smooth.

After dropping the error terms from (6) and (9), we add nonzero constants l (see the matrix D_1 and the bottom of the proof of Theorem 5) and $-h_o$ (see the last column of G) to both sides of (7) and (9) respectively. Then we obtain a square system for the unknown vector

$$\underline{y} = (y(s_1), \ldots, y(s_{n-1}), 1, y(t_1), \ldots, y(t_n))$$

approximating the exact solution

$$\underline{y}^* = (y^*(s_1), \cdots, y^*(s_{n-1}), 1, y^*(t_1), \ldots, y^*(t_n)).$$

The system can be written as

$$M\underline{y} = \begin{bmatrix} D_1 & A \\ G & D_2 \end{bmatrix} \underline{y} = \underline{f} \tag{10}$$

where $\underline{f} = (f_*(s_1), \ldots, f_*(s_{n-1}), C_o + l, f_*(t_1) - h_o, \ldots, f_*(t_n) - h_o)$, and D_1 is a diagonal matrix with

$$\begin{cases} (D_1)_{jj} = a_*(s_j)\rho(s_j) + \frac{\psi_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)}(s_j)} & \text{for } j = 1, \ldots, n-1 \\ (D_1)_{nn} = l \end{cases}$$

We will use $1/w_n$ for l and $1/|b(1)|$ for h_o later (see [6]).

$$D_2 = diag\left(a_*(t_i)\rho(t_i) + \frac{\psi_{n-1}^{(-\alpha,-\beta)}(t_i)}{P_{n-1}^{(-\alpha,-\beta)}(t_i)}\rho^2(t_i)\right) \qquad i = 1, \ldots, n$$

We assume $(D_1)_{ii}$ and $(D_2)_{ii}$ are not zero for all i.

$$A_{ij} = \begin{cases} \frac{w_j}{t_j - s_i} & i \neq n \\ w_j & i = n \end{cases} \qquad j = 1, 2, \ldots, n$$

$$G_{ij} = \begin{cases} \frac{w_j^*}{s_j - t_i}\rho^2(s_j) & j \neq n \\ -h_o & j = n \end{cases} \qquad i = 1, 2, \ldots, n$$

Note that the system is of order 2n. Also we remark that G can be written as the product of two matrices B and D_3.

$$G = BD_3$$

where

$$B_{ij} = \begin{cases} \frac{w_j^*}{s_j - t_i} & j \neq n \\ -h_o & j = n \end{cases} \qquad i = 1, 2, \ldots, n$$

and D_3 is the diagonal matrix with

$$\begin{cases} (D_3)_{jj} = \rho^2(s_j) & j = 1, 2, \ldots, n-1 \\ (D_3)_{nn} = 1 & j = n \end{cases}$$

Let us introduce the matrices:

$$N_1 = \begin{pmatrix} w_1 & & & \\ & \ddots & & \\ & & w_{n-1} & \\ & & & w_n \end{pmatrix}, \; N_2 = \begin{pmatrix} w_1^* & & & \\ & \ddots & & \\ & & w_{n-1}^* & \\ & & & \frac{1}{|b(1)|} \end{pmatrix},$$

$$S = \begin{pmatrix} \frac{1}{t_1 - s_1} & \frac{1}{t_2 - s_1} & \cdots & \frac{1}{t_n - s_1} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{1}{t_1 - s_{n-1}} & \frac{1}{t_2 - s_{n-1}} & \cdots & \frac{1}{t_n - s_{n-1}} \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

With these matrices, the matrices A and B can be rewritten by

$$A = SN_1\,, \quad B = -S^tN_2$$

In this notation, the marix M which is the main matrix in discretized system is expressed in form of two matrices' multiplication.

$$\begin{aligned} M &= \begin{pmatrix} D_1 & S \cdot N_1 \\ -S^t \cdot N_2 \cdot D_3 & D_2 \end{pmatrix} \\ &= \begin{pmatrix} D_1 D_3^{-1} N_2^{-1} & S \\ -S^t & D_2 N_1^{-1} \end{pmatrix} \begin{pmatrix} N_2 D_3 & 0 \\ 0 & N_1 \end{pmatrix}. \end{aligned}$$

Here we use notations P_1 and P_2 for $D_1D_3^{-1}N_2^{-1}$ and $D_2N_1^{-1}$ respectively which are diagonal matrices. Then

$$M = RT$$

where

$$T = \begin{pmatrix} N_2D_3 & 0 \\ 0 & N_1 \end{pmatrix}, \quad R = \begin{pmatrix} P_1 & S \\ -S^t & P_2 \end{pmatrix} \tag{11}$$

Lemma 1. *(i)*

$$\frac{\psi_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)}(s_j)} \sim \rho(s_j)$$

(ii)

$$\frac{\psi_{n-1}^{(-\alpha,-\beta)}(t_i)}{P_{n-1}^{(-\alpha,-\beta)}(t_i)}\rho(t_i) \sim O(1)$$

Proof. ¿From [13], we have

$$\{(\frac{d}{dx})^k P_n^{(\alpha,\beta)}(x)\}_{x=\cos\theta} = \begin{cases} \theta^{-\alpha-k-1/2}O(n^{k-1/2}) & c/n \le \theta \le \pi/2 \\ O(n^{2k+\alpha}) & 0 \le \theta \le c/n. \end{cases}$$

For positive s_j, we have the following estimate:

$$\frac{\psi_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)}(s_j)} \sim \frac{j^{\alpha-1/2}n^{-\alpha}}{j^{-\alpha-1/2}n^{\alpha}} \sim \left(\frac{j}{n}\right)^{2\alpha} \sim \rho(s_j).$$

If s_j is negative, we use β instead of α. This still gives us the same estimate $\rho(s_j)$. Similarly,

$$\frac{\psi_{n-1}^{(-\alpha,-\beta)}(t_i)}{P_{n-1}^{(-\alpha,-\beta)}(t_i)}\rho(t_i) \sim \frac{i^{-\alpha-1/2}n^{\alpha}}{i^{\alpha-1/2}n^{-\alpha}}\left(\frac{i}{n}\right)^{2\alpha} \sim O(1).$$

□

In matrices D_1 and D_2, if the value of $a_*\rho$ is large enough that it exceeds the value of ψ_n/P_n, then we may have positive values in P_1 and P_2 since D_3, N_1 and N_2 are positive diagonal matrices. Also, since $a_*(x)$ doesn't have any zero in $(-1,1)$, we make $a_*(x)$ have positive values without loss of generality.

Hence we may assume P_1 and P_2 are positive diagonal matrices.

Theorem 2. *The system (10) is nonsingular. i.e. The matrix M is nonsingular.*

Proof. It suffices to show R is invertible since $M = RT$ and T is a diagonal matrix with positive entries. R can be decomposed as below.

$$\begin{aligned} R &= \begin{pmatrix} P_1 & S \\ -S^t & P_2 \end{pmatrix} \\ &= \begin{pmatrix} P_1^{\frac{1}{2}} & 0 \\ 0 & P_2^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I_n & P_1^{-\frac{1}{2}}SP_2^{-\frac{1}{2}} \\ -P_2^{-\frac{1}{2}}S^tP_1^{-\frac{1}{2}} & I_n \end{pmatrix} \begin{pmatrix} P_1^{\frac{1}{2}} & 0 \\ 0 & P_2^{\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

Here we observe that the middle matrix Q in the above expression can be written $Q = I + K$ where K is skew-symmetric.

$$\begin{aligned} Q &= \begin{pmatrix} I_n & P_1^{-\frac{1}{2}} S P_2^{-\frac{1}{2}} \\ -P_2^{-\frac{1}{2}} S^t P_1^{-\frac{1}{2}} & I_n \end{pmatrix} \\ &= I_{2n} + K \end{aligned}$$

where

$$K = \begin{pmatrix} 0 & P_1^{-\frac{1}{2}} S P_2^{-\frac{1}{2}} \\ -P_2^{-\frac{1}{2}} S^t P_1^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

Then Q is nonsingular since K is skew-symmetric. Note that K has only pure imaginary eigenvalues. This shows eigenvalues of Q can not be zero. Consequently we have the Theorem hold. $\square$

We have some properties of the matrices A and B if $\alpha + \beta = -1$. In this case $AB = BA = \frac{-1}{b^2(1)} I_n$ i.e. A is the inverse of B if $b^2(1) = -1$. Also $AN_1^{-1}A^t$ becomes a diagonal matrix $\frac{1}{b^2(1)} N_2^{-1}$.

4. Error Analysis

In this section, we will show the convergence of the method proposed and the error bound of this method in uniform norm. Let ϵ_{G_1} be the vector of quadrature errors at the node points t_i and ϵ_{G_2} the one at s_j. Then the system (10) can be rewritten with the errors ϵ_{G_1} and ϵ_{G_2} in (6) and (9) respectively as follows:

$$M\, \underline{y}^* = f^* = (f_1^T - \epsilon_{G_1}^T, f_2^T - \epsilon_{G_2}^T)$$

where y^* is the exact solution and

$$\begin{aligned} f_1^T &= [f_*(s_1), \ldots, f_*(s_{n-1}), C_o + l] \\ f_2^T &= [f_*(t_1) - h_o, \ldots, f_*(t_n) - h_o]. \end{aligned}$$

Let us define the error vector

$$\underline{e} = \underline{y}^* - \underline{y}.$$

We have then

$$M\underline{e} = (-\epsilon_{G_1}^T, -\epsilon_{G_2}^T)^T$$

This system can be written as

$$RT\underline{e} = \underline{\epsilon} \qquad \text{where } \underline{\epsilon} = \begin{pmatrix} -\epsilon_{G_1}^T \\ -\epsilon_{G_2}^T \end{pmatrix}.$$

Hence we have the exact error expression by Theorem 2 as follows:

$$\underline{e} = T^{-1} R^{-1} \underline{\epsilon}.$$

Taking the Euclidean norms, we have the estimate

$$\|e\|_2 \le \|T^{-1}\|_2 \, \|R^{-1}\|_2 \, \|\epsilon\|_2. \tag{12}$$

Lemma 3. *R is nonsysmetric positive definite.*

Proof. For any nonzero $x = (x_1^t, x_2^t)^t \in \mathbb{R}^{2n}$,

$$\begin{aligned} x^t R x &= (x_1^t, x_2^t)\begin{pmatrix} P_1 & S \\ -S^t & P_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1^t, x_2^t)\begin{pmatrix} P_1 x_1 + S x_2 \\ -s^t x_1 + P_2 x_2 \end{pmatrix} \\ &= x_1^t P_1 x_1 + x_1^t S x_2 - x_2^t S^t x_1 + x_2^t P_2 x_2 \\ &= P_1 \|x_1\|^2 + P_2 \|x_2\|^2 \ > \ 0 \end{aligned}$$

since P_1 and P_2 are positive diagonal matrices and $x_1^t S x_2 - x_2^t S^t x_1 = 0$ as below.

$$x_1^t S x_2 = (Sx_2, x_1) = (x_1, Sx_2) = (Sx_2)^t x_1 = x_2^t S^t x_1$$

□

Now let us define the symmetric part of matrix R by

$$U = \frac{1}{2}(R + R^t) = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Note that U is symmetric positive definite by Lemma 3.

Lemma 4.

$$\|R^{-1}\|_2 \ \le \ \|U^{-1}\|_2$$

Proof. It suffices to show

$$\sigma_n(R) \ge \lambda_n(U)$$

where $\sigma_n(R)$ is the smallest singular value of R and $\lambda_n(U)$ is the smallest eigenvalue of U. Since U is symmetric positive definite, $\lambda_n(U)$ is positive i.e.

$$\sigma_n(R) \ge \lambda_n(U) > 0.$$

This shows

$$\frac{1}{\sigma_n(R)} \le \frac{1}{\lambda_n(U)}.$$

Consequently, this lemma holds. For any $x \in \mathbb{R}^{2n}$ with $\|x\|_2 = 1$,

$$x^t U x = \frac{1}{2}(x^t R x + x^t R^t x) = x^t R x \le \|x\|_2 \|Rx\|_2.$$

By Courant-Weyl theorem,

$$\begin{aligned} \lambda_n(U) &= \min_{w_1,\ldots,w_{n-1} \in \mathbb{R}^{2n}} \ \max_{\substack{x \in \mathbb{R}^{2n} \\ x \perp w_1,\ldots,w_{n-1}}} x^t U x \\ &\le \min_{w_1,\ldots,w_{n-1} \in \mathbb{R}^{2n}} \ \max_{\substack{x \in \mathbb{R}^{2n} \\ x \perp w_1,\ldots,w_{n-1}}} ||Rx||_2 \\ &= \sigma_n(R) \end{aligned}$$

□

By using Lemma 4, the error bound (12) becomes

$$\|e\|_2 \le \|T^{-1}\|_2 \ \|U^{-1}\|_2 \ \|\epsilon\|_2. \tag{13}$$

Hence we have the error in uniform norm in the following Theorem.

Theorem 5.

$$\|e\|_\infty \le C_1 n^{1/2} \|\epsilon\|_\infty$$

where

$$\|\epsilon\|_\infty = \max(\|\epsilon_{G_1}\|_\infty, \|\epsilon_{G_2}\|_\infty).$$

Proof. Since T and U are digonal matrices, from (13),

$$\begin{aligned} \|e\|_2 &\le \|T^{-1}\|_2 \ \|U^{-1}\|_2 \ \|\epsilon\|_2 \\ &= \|T^{-1}\|_\infty \ \|U^{-1}\|_\infty \ \|\epsilon\|_2 \\ &\le \|T^{-1}\|_\infty \ \|U^{-1}\|_\infty \ n^{\frac{1}{2}} \ \|\epsilon\|_\infty . \end{aligned}$$

Here we can get the uniform norm of T^{-1} which is expessed in (11). From [13], we have

$$w_i \ = \ O(n^{-1}) \qquad \text{and} \qquad w_j^* \cdot \rho^2(s_j) \ \sim \ w_j \tag{14}$$

since

$$\begin{aligned} w_j^* \cdot \rho^2(s_j) &\sim j^{-2\alpha+1} \cdot n^{2\alpha-2} \cdot \left(\frac{j}{n}\right)^{4\alpha} \\ &= j^{2\alpha+1} \cdot n^{-2\alpha-2} \\ &\sim w_j \ . \end{aligned}$$

This gives us

$$\|T^{-1}\|_\infty \ = \ O(n) \tag{15}$$

On the other hand,

$$U^{-1} \ = \ \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2^{-1} \end{pmatrix} \ = \ \begin{pmatrix} N_2 D_3 D_1^{-1} & 0 \\ 0 & N_1 D_2^{-1} \end{pmatrix}$$

By Lemma 1, $(D_2)_{ii} \ \sim \ \rho(t_i)$, thus for the positive t_i (If we use the negative t_i, then we replace α by β.),

$$\begin{aligned} (N_1 D_2^{-1})_{ii} &\sim w_i \cdot \rho^{-1}(s_j) \\ &\sim i^{2\alpha+1} \cdot n^{-2\alpha-2} \cdot \left(\frac{i}{n}\right)^{-2\alpha} \\ &= i \cdot n^{-2}. \end{aligned}$$

This gives us the following,

$$\|N_1 D_2^{-1}\|_\infty \ = \ O(n^{-1}). \tag{16}$$

Also, for $i \neq n$,

$$(N_2 D_3 D_1^{-1})_{ii} \ \sim \ w_i^* \cdot \rho^2(s_i) \cdot \rho^{-1}(s_i) \ = \ w_i^* \cdot \rho(s_i) \ \sim \ w_i \cdot \rho^{-1}(s_i)$$

by (14). In case $i = n$,

$$(N_2 D_3 D_1^{-1})_{nn} \ = \ |b(1)|^{-1} \cdot \frac{1}{l} \ = \ |b(1)|^{-1} \cdot w_n \ = \ O(n^{-1}).$$

Similarly,

$$\|N_2 D_3 D_1^{-1}\|_\infty \;=\; O(n^{-1}). \tag{17}$$

¿From (16) and (17),

$$\|U^{-1}\|_\infty \;=\; O(n^{-1}). \tag{18}$$

Here we have the following bound,

$$\|e\|_\infty \;\leq\; C_3 \cdot n^{\frac{1}{2}} \cdot \|\epsilon\|_\infty$$

since $\|T^{-1}\|_\infty \|U^{-1}\|_\infty \;\leq\; C_4$ from (15) and (18). □

To get the error bound of the system, we need to know the error of Hunter's method. It can be obtained from [2, 3, 16]. Now we have the error for Hunter's method as follows.

Lemma 6. *If $y^{*(m)} \in H_\mu$ with $m + \mu \geq 1$, then for $x \in (-1, 1)$*

$$|\epsilon_G| = O\left(\frac{\ln n}{n^{\mu - 2\nu}}\right)$$

where ν is any positive number such that $2\nu < \mu$.

Lemma 6 and Theorem 5 let us have the convergence of the proposed method and its convergence rate is given by the following Theorem.

Theorem 7. *$y^{*(m)} \in H_\mu$ with $m + \mu \geq 1$, the following estimate holds*

$$\|e\|_\infty \leq C_5 \; n^{-(m+\mu-1/2-2\nu-\varepsilon)}$$

with $\varepsilon > 0$ arbitrarily small.

In this procedure, we need $y^* \in H^1$ to obtain the convergence [11, 3]. If we choose Elliott-Paget method [3] instead of Hunter's, and proceed in a similar way, we have the convergence and its convergence rate is given as below

Corollary 8. *If $y^{*(m)} \in H_\mu$, with $m + \mu > 1/2$,*

$$\|e\|_\infty \leq C_6 \; n^{-(m+\mu-1/2-2\nu-\varepsilon)}.$$

Finally, we construct the approximate solution represented by the Lagrange interpolatory polynomial $P_{2n-1}(x)$ on the nodes $\{t_1, s_1, \cdots, s_{n-1}, t_n\}$,

$$P_{2n-1}(x) \;=\; \frac{1}{2}\left[\sum_{i=1}^{n} \; y(t_i)\frac{P_n^{(\alpha,\beta)}(x)}{(x - t_i)\; P_n^{(\alpha,\beta)'}(t_i)} + \sum_{j=1}^{n-1} \; y(s_j)\frac{P_{n-1}^{(-\alpha,-\beta)}(x)}{(x - s_j)\; P_{n-1}^{(-\alpha,-\beta)'}(s_j)}\right].$$

Let $P^*_{2n-1}(x)$ be the polynomial of degree $2n-1$ interpolating on the exact values of $y^*(x)$. Then

$$\begin{aligned} |P^*_{2n-1}(x) \ - \ P_{2n-1}(x)| \ &\le \ \frac{1}{2}\left[\sum_{i=1}^{n} \ |y^*(t_i) - y(t_i)| \left| \frac{P_n^{(\alpha,\beta)}(x)}{(x-t_i)\ P_n^{(\alpha,\beta)'}(t_i)} \right| \right. \\ &\qquad \left. + \sum_{j=1}^{n-1} \ |y^*(s_j) - y(s_j)| \left| \frac{P_{n-1}^{(-\alpha,-\beta)}(x)}{(x-s_j)\ P_{n-1}^{(-\alpha,-\beta)'}(s_j)} \right| \right] \\ &\le \ \|e\|_\infty \cdot \Lambda_p \end{aligned}$$

where Λ_p is the Lebesgue constant [13] and

$$\Lambda_p = O(\log n).$$

Consequently, we have an estimate between the exact solution and the approximate solution as follows.

$$\begin{aligned} \|y^* - P_{2n-1}\|_\infty \ &\le \ \|y^* - P^*_{2n-1}\|_\infty \ + \ \|P^*_{2n-1} - P_{2n-1}\|_\infty \\ &\le \ C_7\omega(y^*, \frac{1}{n}) \ + \ C_8\|e\|_\infty \ln n \end{aligned}$$

where ω is the modulus of continuity. This ensures the rate convergence under the result of Theorem 7.

Theorem 9. *If* $y^{*(m)} \in H_\mu$ *with* $m + \mu \ge 1$*, then*

$$\|y^* - P_{2n-1}\|_\infty \ \le \ C_9 \cdot n^{-(m+\mu-1/2-2\nu-\varepsilon)}$$

The above result can be strengthened with the Elliott-Paget method i.e. we have a weaker condition such as $m + \mu > 1/2$. The rates of convergence of two methods are almost same even if they have different restrictions on the smoothness of the function y^*.

References

[1] M.M. Chawla, T. R. Ramakrishnan, (1974), Modified Gauss-Jacobi quadrature formulas for the numerical evaluation of Cauchy type singular integrals, *BIT 14*, pp. 14-21.

[2] D. Elliott, D. F. Paget, (1975), On the convergence of a quadrature rule for evaluating certain Cauchy principal value integrals, *Numer. Math. 23*, pp. 311-319.

[3] D. Elliott, D. F. Paget, (1979), Gauss type quadrature rules for Cauchy principal value integrals, *Math of Computation, Vol 33*, pp. 301-309.

[4] D. Elliott, (1979), The numerical solution of singular integral equations over (-1, 1), *SIAM J. Numer. Anal., 16*, pp. 115-134.

[5] D. Elliott, (1982), The numerical treatment of singular integral equations — a review, in *Treatment of Integral Equations by Numerical Methods*, C.T.H. Baker, G.F. Miller (Editors), Acad. Press., New York.

[6] A. Gerasoulis, R. P. Srivastav, (1982), On the solvability of singular integral equations via Gauss-Jacobi quadrature, *Intern. J. Computer Math., Vol. 12*, pp. 59-75.

[7] A. Gerasoulis, (1986), The singular value decomposition of the Gauss-Chebyshev and Lobatto-Chebyshev methods for Cauchy singular integral equations, *Comp. and Maths with Appls., 12A*, pp. 895-907.

[8] D. B. Hunter, (1972), Some Gauss-type formulae for the evaluation of Cauchy principal values of integrals, *Numer. Math. 19*, pp. 419-424.
[9] A. I. Kalandiya, (1975), *Mathematical Methods of two-dimensional elasticity*, Moscow, Mir Publishers.
[10] N.I. Muskhelishvili, (1953), *Singular integral equations*, P. Noordhoff, Groningen, Holland.
[11] P. Rabinowitz, (1984), On the convergence and divergence of Hunter's method for Cauchy principal Value integrals, in *Numerical solution of singular integral equations*, A. Gerasoulis, R. Vinchnevetsky (Editors), proceedings of IMACS, published by IMACS.
[12] R.P. Srivastav, Fenggang Zhang, (1991), Solving Cauchy singular integral equations by using general quadrature-collocation nodes, *Computers Math. Applic. Vol. 21, No. 9*, pp. 59-71.
[13] G. Szego, (1975), *Orthogonal Polynomials*, AMS Colloquium Publications, vol. XXIII, 4th Ed. Philadelphia.
[14] P.S. Theocaris, G. Tsamasphyros, (1979), Numerical solution of systems of singular integral equations with variable coefficients, *Applicable Analysis, Vol. 9*, pp. 37-52.
[15] G. Tsamasphyros, P.S. Theocaris, (1981), Equivalence and convergence of direct and indirect methods for the numerical solution of singular integral equations, *Computing 27*, pp. 71-80.
[16] G. Tsamasphyros, P.S. Theocaris, (1983), On the convergence of some quadrature rules for Cauchy-value and finite-part integrals, *Computing 31*, pp. 105-114.

Department of Mathematics
SungKyunKwan University
email: skim@yurim.skku.ac.kr