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# OPTIMALITY FOR MULTIOBJECTIVE FRACTIONAL VARIATIONAL PROGRAMMING

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ABSTRACT. We consider a multiobjective fractional variational programming problem (P) involving vector valued functions. By using the concept of proper efficiency, a relationship between the primal problem and parametric multiobjective variational problem is indicated.

## 1. Introduction and problems

Programs with several conflicting objectives have been extensively studied in literatures [1]-[11]. Introducing the concept of proper efficiency of solutions, Geoffrion[5] proved an equivalence between a multiobjective program with convex functions and a related parametric objective program. Using this equivalence, optimality and duality for multiobjective variational problems have been of much interest in recent years, and contributions have been made to its development. Bector and Husain[2] formulated a dual program for a multiobjective variational program having properly efficient solutions. Also, using parametric equivalence, Bector and Husain[1] formulated a dual program for a multiobjective fractional program having continuously differentiable convex functions. Further, Mishra and Mukherjee[6] considered the duality of multiobjective fractional variational problems by relating the primal problem to a parametric multiobjective variational problem.

Motivated by the above results, in this paper we propose studying optimality for multiobjective fractional variational problems having properly efficient solutions.

We consider a multiobjective fractional variational programming problem (P) involving vector valued functions.

$$\text{(P)} \qquad \text{Minimize } \frac{\int_a^b f(t,x(t),\dot{x}(t))dt}{\int_a^b g(t,x(t),\dot{x}(t))dt} = \left[\frac{\int_a^b f^1(t,x(t),\dot{x}(t))dt}{\int_a^b g^1(t,x(t),\dot{x}(t))dt},\cdots,\frac{\int_a^b f^p(t,x(t),\dot{x}(t))dt}{\int_a^b g^p(t,x(t),\dot{x}(t))dt}\right]$$

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subject to

$$x(a) = \alpha, \;\; x(b) = \beta \tag{1}$$
$$h^j(t, x(t), \dot{x}(t)) \leq 0, \;\; t \in I, j = 1, 2, \cdots, m,$$
$$x \in S, \quad \dot{x} \in C(I, R^n). \tag{2}$$

We assume that $g^i(t, x(t), \dot{x}(t)) > 0$ and $f^i(t, x(t), \dot{x}(t)) \geq 0$ whenever $g^i(x)$ is not linear for all $i = 1, 2, \cdots, p$.

To optimize (P) is to find properly efficient solutions.

Geoffrion[5] introduced the definition of the properly efficient solution in order to eliminate the efficient solutions causing unbounded trade-offs between objective functions.

Corresponding to (P), we consider the following parametric vector variational problem ($\mathrm{P}_v$).

$$(\mathrm{P}_v) \qquad \text{Minimize } \Big[ \int_a^b \{f^1(t, x(t), \dot{x}(t)) - v_1 g^1(t, x(t), \dot{x}(t))\} dt$$
$$, \cdots, \int_a^b \{f^p(t, x(t), \dot{x}(t)) - v_p g^p(t, x(t), \dot{x}(t)\} dt \Big]$$

subject to

$$x(a) = \alpha, \;\; x(b) = \beta, \tag{1}$$
$$h^j( \; t, x(t), \dot{x}(t) \; ) \leq 0, \;\; t \in I, \;\; j = 1, 2, \cdots, m,$$
$$x \in S, \;\; \dot{x} \in C(I, R^n). \tag{2}$$

In this paper, we prove that (P) and ($\mathrm{P}_v$) have equivalent properly efficient solutions. We can obtain optimality and duality for (P) by using of this equivalent relation.

We give some definitions and results from [2] and [7], which are used subsequently in our later results.

Let $I = [a, b]$ be a real interval and $f = (f^1, \cdots, f^p) \; : I \times R^n \times R^n \to R^p$, $g = (g^1, \cdots, g^p) \; : I \times R^n \times R^n \to R^p$ and $h = (h^1, \cdots, h^m) \; : I \times R^n \times R^n \to R^m$ be continuously differentiable functions. In order to consider $f^i(t, x(t), \dot{x}(t))$, where $x : I \to R^n$ with derivative $\dot{x}$, denote the partial derivative of $f^i$ with respect to $t, x$ and $\dot{x}$ respectively, by $f^i_t, f^i_x$ and $f^i_{\dot{x}}$ such that

$$f^i_x = (\frac{\partial f^i}{\partial x_1}, \frac{\partial f^i}{\partial x_2}, \cdots, \frac{\partial f^i}{\partial x_n}) \;\; , \;\; f^i_{\dot{x}} = (\frac{\partial f^i}{\partial \dot{x}_1}, \frac{\partial f^i}{\partial \dot{x}_2}, \cdots, \frac{\partial f^i}{\partial \dot{x}_n}).$$

Similary, we write the partial derivatives of the vector functions $g$ and $h$ using matrices $p \times n$ and $m \times n$, respectively.

Let $C(I,R^n)$ denote the space of piecewise smooth functions $x$ with norm $\| x \| = \| x \|_\infty + \| Dx \|_\infty$, where the differentiation operator $D$ is given by

$$u = Dx \quad \Leftrightarrow \quad x(t) = \alpha + \int_\alpha^t u(s)ds,$$

where $\alpha$ is a given boundary value.

Therefore, $D = \frac{d}{dt}$ except at discontinuties.

Let $S \subseteq R^n$ be open.

Let X the set of all feasible solutions of (P) be given by

$$\begin{aligned} X \ = \ & \{x \in C(I,R^n) \mid x(a) = \alpha, \ x(b) = \beta, \ h^j(t,x(t),\dot{x}(t)) \leq 0, \\ & t \in I, \ j = 1,2,\cdots,m \ \}. \end{aligned}$$

In the sequel we shall always denote the set $\{1,2,\cdots,p\}$ and $\{1,2,\cdots,m\}$ by $\bar{p}$ and $\bar{m}$ respectively.

Definition 1. A feasible solution $x^*$ of (P) is an efficient solution of (P) if there exist no other feasible $x$ for (P) such that

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} \leq \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} \text{ for all } i \in \bar{p} \tag{3}$$

and

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} < \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \text{ for some } j \in \bar{p}. \tag{4}$$

By eliminating efficient solutions causing unbounded trade-off between objective functions, we can define the properly efficient solutions as follows.

Definition 2[5]. A feasible solution $x^*$ of (P) is a properly efficient solution of (P) if it is efficient and if there exists a scalar $M > 0$ such that, for each $i$,

$$\frac{\dfrac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} - \dfrac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt}}{\dfrac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} - \dfrac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}} \leq M$$

for some j such that

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} > \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}$$

whenever $x$ is feasible for (P) and

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} < \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt}.$$

An efficient point that is not properly efficient is said to be improperly efficient. Thus for $x^*$ to be improperly efficient means that for every scalar $M > 0$(no matter how large) there is feasible point $x$ and an $i$ such that

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} < \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt}$$

and

$$\frac{\frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} - \frac{\int_a^b f^i(t,x^(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt}}{\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} - \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}} > M$$

for all j such that

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} > \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}.$$

## 2. Main Result

The following theorem connects (P) and $(\mathrm{P}_v)$ with $v = v^*$.

THEOREM 3 $x^*$ is a properly efficient solution of (P) if and only if there exists

$$v_j^* = \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt},$$

for some $j \in \bar{p}$ such that $x^*$ is a properly efficient solution of $(\mathrm{P}_v)$ with $v = v^*$.

*Proof.* Let $x^*$ be a properly efficient solution of (P) and let

$$v_j^* = \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t)dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \text{ for some } j \in \bar{p}. \tag{5}$$

If $x^*$ is not an efficient solution of $(\mathrm{P}_v)$ with $v = v^*$, then there exists a feasible solution $x$ of $(\mathrm{P}_v)$ with $v = v^*$, such that

$$\int_a^b \{f^i(t,x(t),\dot{x}(t)) - v_i^* g_i(t,x(t),\dot{x}(t))\}dt$$
$$\leq \int_a^b \{f^i(t,x^*(t),\dot{x}^*(t)) - v_i^* g^i(t,x^*(t),\dot{x}^*(t))\}dt \text{ for all } i \in \bar{p}$$

and

$$\int_a^b \{f^j(t,x(t),\dot{x}(t)) - v_j^* g^j(t,x(t),\dot{x}(t))\}dt$$
$$< \int_a^b \{f^j(t,x^*(t),\dot{x}^*(t)) - v_j^* g^j(t,x^*(t),\dot{x}(t)^*)\}dt \text{ for some } j \in \bar{p}.$$

It follows that

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} \leq \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} \text{ for all } i \in \bar{p} \tag{6}$$

and

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} < \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \text{ for some } j \in \bar{p}. \tag{7}$$

Contradicting the efficiency of $x^*$ in (P). Hence $x^*$ is an efficient solution of ($\mathrm{P}_v$) with $v = v^*$.

Now we shall show $x^*$ is a properly efficient solution of ($\mathrm{P}_v$) with $v = v^*$. If $x^*$ is not properly efficient for ($\mathrm{P}_v$) with $v = v^*$, then, for every sufficiently large scalar $M > 0$, there is $x \in \mathrm{X}$ and an $i$ such that

$$\int_a^b \{f^i(t,x(t),\dot{x}(t)) - v_i^* g^i(t,x(t),\dot{x}(t))\}dt < 0 \tag{8}$$

and

$$\frac{\int_a^b \{f^i(t,x^*(t),\dot{x}^*(t)) - v_i^* g^i(t,x^*(t),\dot{x}(t)^*)\}dt - \int_a^b \{f^i(t,x(t),\dot{x}(t)) - v_i^* g^i(t,x(t),\dot{x}(t))\}dt}{\int_a^b \{f^j(t,x(t),\dot{x}(t)) - v_j^* g^j(t,x(t),\dot{x}(t))\}dt - \int_a^b \{f^j(t,x^*(t),\dot{x}^*(t)) - v_j^* g^j(t,x^*(t),\dot{x}^*(t))\}dt}$$

$$> M \tag{9}$$

for all $j$ such that

$$\int_a^b \{f^j(t,x(t),\dot{x}(t)) - v_j^* g^j(t,x(t),\dot{x}(t))\}dt > 0 \tag{10}$$

i.e.,

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} < \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} \tag{8$'$}$$

$$\frac{-\int_a^b f^i(t,x(t),\dot{x}(t))dt + v_i^* \int_a^b g^i(t,x(t),\dot{x}(t))dt}{\int_a^b f^j(t,x(t),\dot{x}(t))dt - v_j^* \int_a^b g^j(t,x(t),\dot{x}(t))dt} \quad > \quad M \tag{9'}$$

for all $j$ such that

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} \quad > \quad \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \tag{10'}$$

Now (9') can be rewritten as

$$\frac{\frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} - \frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt}}{\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} - \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}} \quad > \quad M.$$

So $(8'), (9')$ and $(10')$ imply that $x^*$ is not properly efficient for (P).
Hence $x^*$ is properly efficient in $(P_v)$ with $v = v^*$.

Conversely, let $x^*$ be a properly efficient solution of $(P_v)$ with $v = v^*$ where

$$v_j^* \quad = \quad \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \quad j \in \bar{p}. \tag{5}$$

then we shall show that $x^*$ is properly efficient for (P).

If $x^*$ is not an efficient solution of (P), then there exists a feasible solution $x$ for (P) such that

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} \quad \leq \quad \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} \text{ for all } i \in \bar{p} \tag{6}$$

and

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} \quad < \quad \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \text{ for some } j \in \bar{p}. \tag{7}$$

Now (5) together with (6) and (7) contradict the efficiency of $x^*$ in $(P_v)$ with $v = v^*$
.

Thus $x^*$ is an efficient solution of (P).

Now we shall show that $x^*$ is a properly efficient solution of (P).

If $x^*$ is not properly efficient for (P), then, there is an $x \in X$ and an $i \in \bar{p}$ such that

$$\frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt} \quad < \quad \frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t))dt} \tag{11}$$

and

$$\frac{\frac{\int_a^b f^i(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^i(t,x^*(t),\dot{x}^*(t)\ \ )dt} - \frac{\int_a^b f^i(t,x(t),\dot{x}(t))dt}{\int_a^b g^i(t,x(t),\dot{x}(t))dt}}{\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} - \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt}} > M. \tag{12}$$

for all $M > 0$ and for all $j$ such that

$$\frac{\int_a^b f^j(t,x(t),\dot{x}(t))dt}{\int_a^b g^j(t,x(t),\dot{x}(t))dt} > \frac{\int_a^b f^j(t,x^*(t),\dot{x}^*(t))dt}{\int_a^b g^j(t,x^*(t),\dot{x}^*(t))dt} \tag{13}$$

i.e.,

$$\int_a^b f^i(t,x(t),\dot{x}(t))dt - v_i^* \int_a^b g^i(t,x(t),\dot{x}(t))dt < 0. \tag{11$'$}$$

and

$$\frac{-\int_a^b f^i(t,x(t),\dot{x}(t))dt + v_i^* \int_a^b g^i(t,x(t),\dot{x}(t))dt}{\int_a^b f^j(t,x(t),\dot{x}(t))dt - v_j^* \int_a^b g^j(t,x(t),\dot{x}(t))dt} > M \tag{12$'$}$$

for all $j$ such that

$$\int_a^b f^j(t,x(t),\dot{x}(t))dt - v_j^* \int_a^b g^j(t,x(t),\dot{x}(t))dt > 0. \tag{13$'$}$$

So $(11'),(12')$ and $(13')$ imply that $x^*$ is not properly efficient for $(P_v)$ with $v = v^*$. Hence $x^*$ is properly efficient in (P).

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