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L_p and $W^{1,p}$ Error Estimates for First Order GDM on One-Dimensional Elliptic and Parabolic Problems

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Abstract

In this paper, we consider first order generalized difference scheme for the two-point boundary value problem and one-dimensional second order parabolic type problem. The optimal error estimates in L_p and $W^{1,p}$ $(2 \le p \le \infty)$ as well as some superconvergence estimates in $W^{1,p}$ $(2 \le p \le \infty)$ are obtained. The main results in this paper perfect the theory of GDM.

1 Introduction

We first consider the two-point boundary problem

$$\begin{cases} (a) \quad Lu \equiv -\dfrac{d}{dx}(p\dfrac{du}{dx}) = f, \quad a < x < b, \\ (b) \quad u(a) = 0, \quad u(b) = 0, \end{cases} \tag{1.1}$$

where $p = p(x) \ge p_{\min} > 0$, $p \in C^1(I)$, $f \in L^2(I)$, $I = [a,b]$. Secondly, we consider one-dimentional second order parabolic type problem

$$\begin{cases} (a) \quad \dfrac{\partial u}{\partial t} - \dfrac{\partial}{\partial x}(p\dfrac{\partial u}{\partial x}) = f(x,t), \quad (x,t) \in (a,b) \times (0,T], \\ (b) \quad u(x,0) = 0, \quad x \in [a,b], \\ (c) \quad u(a,t) = 0, \quad u(b,t) = 0, \quad t \in [0,T], \end{cases} \tag{1.2}$$

where $p = p(x) \ge p_{\min} > 0$, $p \in C^1(I)$, $f \in L^2(I)$, $I = [a,b]$. In the past several decades, Li and other authors did extensive and deep research on the theory and application of generalized difference methods (GDM for short), including constructing first order or higher order difference schemes on elliptic, parabolic and hyperbolic equations, establishing the optimal Sobolev norm estimates of errors, and applying GDM to underground fluid, electromagnetic field and other fields. Theoretical researches and realistic computations show that GDM not only keep the computational simplicity of

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difference methods, but also enjoy the accuracy of finite element methods. See [4] and [7] for more details.

Since the time of studying GDM is only several decades and establishing the error estimates is very difficult, the theory of GDM is not perfect. For example, the optimal order H^1, L_2 and maximum norm error estimates to problem (1.1) and (1.2) have been obtained (See [4], [5], [6], [7]), but the $W^{1,p}$ and L_p $(2 < p < \infty)$ norm error estimates as well as some superconvergence estimates have not been derived before. We shall do this work in this paper. Moreover, by using Green functions, we will reduce the demand of the smoothness of u in the error estimates of maximum norms, and get a perfect statement combining the case of $2 \leq p \leq \infty$.

This paper is organized in the following way. In section 2 we do some preparations, including introducing the partitions of the interval I, the trial and test function spaces, Green functions and some lemmas which are essential in our analysis. We consider problem (1.1) and establish optimal error estimates of $u - u_h$ in $W^{1,p}(I)$ and $L_p(I)$ $(2 \leq p \leq \infty)$ as well as the superconvergence estimate of $\tilde{u} - u_h$ in $W^{1,p}(I)$ $(2 \leq p \leq \infty)$ in section 3. Section 4 deals with the problem (1.2) and we will obtain the optimal error estimates of $u - u_h$ in $W^{1,p}(I)$ and $L_p(I)$ $(2 \leq p \leq \infty)$ in addition to the superconvergence estimates of $\tilde{u} - u_h$ and $R_h^* u - u_h$ in $W^{1,p}(I)$ $(2 \leq p \leq \infty)$.

2 Preparations

Let $U = H_0^1(I) \equiv \{v \in H^1(I),\ v(a) = v(b) = 0\}$. Then the weak form of (1.1) is to find $u \in U$ such that

$$a(u, v) = (f, v), \qquad \forall v \in U, \tag{2.1}$$

where

$$a(u, v) = \int_a^b p u' v' dx, \quad (f, v) = \int_a^b f v dx.$$

We first define the partition T_h of the interval $I = [a, b]$ with nodes x_i, $i = 1, 2, \cdots, n$,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Let $h_i = x_i - x_{i-1}$ denote the length of the element $I_i = [x_{i-1}, x_i]$, $h = \max\limits_{1 \leq i \leq n} h_i$ and let the partition T_h of I be regular, that is, there exists a constant $\mu > 0$ such that

$$h_i \geq \mu h, \quad i = 1, 2, \cdots, n. \tag{2.2}$$

Corresponding to the partition T_h, we then introduce its dual partition T_h^* with nodes $x_{i+\frac{1}{2}}$, $i = 0, \cdots, n-1$,

$$a = x_0 < x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{n-\frac{1}{2}} < x_n = b.$$

$I_0^* = [x_0, x_{\frac{1}{2}}]$, $I_j^* = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ $(j = 1, 2, \cdots, n-1)$, and $I_n^* = [x_{n-\frac{1}{2}}, x_n]$ are said to be dual elements, here

$$x_{j-\frac{1}{2}} = \frac{1}{2}(x_{j-1} + x_j), \qquad 1 \le j \le n.$$

Corresponding to the partition T_h, we choose the trial function space U_h be the space of continuous piecewise linear functions, and $U_h = \text{span}\{\phi_i(x), 1 \le i \le n-1\}$, The basis function ϕ_i corresponding to the node x_i is

$$\phi_i(x) = \begin{cases} 1 - h_i^{-1}|x - x_i|, & x_{i-1} \le x \le x_i, \\ 1 - h_{i+1}^{-1}|x - x_i|, & x_i \le x \le x_{i+1}, \\ 0, & \text{otherwise}, \end{cases} \qquad i = 1, 2, \cdots, n-1,$$

so, any $u_h \in U_h$ can be expressed uniquely in the following way:

$$u_h(x) = \sum_{i=1}^{n-1} u_i \phi_i(x),$$

where $u_i = u_h(x_i)$, and on each element I_i, $i = 1, 2, \cdots, n$,

$$u_h'(x) = \frac{u_i - u_{i-1}}{h_i}, \quad x_{i-1} \le x \le x_i. \tag{2.3}$$

Corresponding to the partition T_h^*, let the test function space V_h be the space of piecewise constant functions. Then the basis of V_h may be taken to be characteristic functions of elements I_j^*,

$$\psi_j(x) = \begin{cases} 1, & x \in I_j^*; \\ 0, & \text{otherwise}, \end{cases} \qquad j = 1, 2, \cdots, n-1.$$

and each $v_h \in V_h$ can be expressed uniquely in the following way:

$$v_h(x) = \sum_{j=1}^{n-1} v_j \psi_j(x).$$

Obviously

$$U_h(x) \subset U \cap W^{1,\infty}(I), \quad V_h \subset L_2(I).$$

Let's define, for any $u_h \in U_h$ and $v_h \in V_h$, a bilinear form as follows

$$\begin{aligned} (a) \quad a^*(u_h, v_h) &= \sum_{j=1}^{n-1} v_j a^*(u_h, \psi_j), \\ (b) \quad a^*(u_h, \psi_j) &= p_{j-\frac{1}{2}} u_h'(x_{j-\frac{1}{2}}) - p_{j+\frac{1}{2}} u_h'(x_{j+\frac{1}{2}}) \\ &\equiv p_{j-\frac{1}{2}} \frac{u_j - u_{j-1}}{h_j} - p_{j+\frac{1}{2}} \frac{u_{j+1} - u_j}{h_{j+1}}, \end{aligned} \tag{2.4}$$

where $u_j = u_h(x_j)$, $v_j = v_h(x_j)$, $p_{j-\frac{1}{2}} = p(x_{j+\frac{1}{2}})$, and $j = 1, 2, \cdots, n-1$.

For numerical analysis, we need to introduce the interpolation operators $\Pi_h : U \to U_h$, defined by

$$\Pi_h w = \sum_{i=1}^{n-1} w(x_i)\phi_i(x), \quad \forall w \in U,$$

and $\Pi_h^* : U \to V_h$, defined by

$$\Pi_h^* w = \sum_{j=1}^{n-1} w(x_j)\psi_j(x), \quad \forall w \in U.$$

Using the theory of Sobolev's interpolation, we have

$$|w - \Pi_h w|_{m,p} \leq Ch^{k-m}|w|_{k,p}, \quad m = 0, 1, \quad k = 1, 2, \quad 1 \leq p \leq \infty, \tag{2.5}$$

$$|\Pi_h w|_{m,p} \leq C|w|_{m,p}, \quad m = 0, 1, \quad 2 \leq p \leq \infty, \tag{2.6}$$

where $|\cdot|_{m,p}$ and $\|\cdot\|_{m,p}$ stand for the semi-norm and norm of the Sobolev space $W^{m,p}(I)$ respectively, $|\cdot|_m$ and $\|\cdot\|_m$ stand for the semi-norm and norm of the Sobolev space $H^m(I) = W^{m,2}(I)$ respectively, and C is a positive constant independent of h.

Noting that for any $u_h \in U_h$, we have, by (2.4),

$$|u_h|_1 = [\int_a^b |u_h'|^2 dx]^{\frac{1}{2}} = [\sum_{i=1}^{n} \frac{(u_i - u_{i-1})^2}{h_i}]^{\frac{1}{2}}. \tag{2.7}$$

Define the discrete L^2-norm

$$\|u_h\|_{0,h} = [\sum_{i=1}^{n} h_i(u_{i-1}^2 + u_i^2)]^{\frac{1}{2}}, \quad u_h \in U_h. \tag{2.8}$$

Then we can easily prove the following lemmas (See [4], [7]).

Lemma 2.1 There exist positive constants C_1 and C_2 such that

$$C_1\|u_h\|_{0,h} \leq \|u_h\|_0 \leq C_2\|u_h\|_{0,h}, \quad \forall u_h \in U_h. \tag{2.9}$$

Lemma 2.2 For any $u_h \in U_h$, the norms $\|\Pi_h^* u_h\|$ and $|||u_h|||_0 = (u_h, \Pi_h^* u_h)^{\frac{1}{2}}$ are equivalent to the L_2 norm $\|u_h\|_0$.

Lemma 2.3 For any u_h, $w_h \in U_h$, one has

$$a^*(u_h, \Pi_h^* w_h) = a^*(w_h, \Pi_h^* u_h), \tag{2.10}$$

$$(u_h, \Pi_h^* w_h) = (w_h, \Pi_h^* u_h). \tag{2.11}$$

Lemma 2.4 The bilinear form $a^*(\cdot, \Pi_h^*\cdot)$ is bounded over $U_h \times U_h$, that is, there exists a constant $M > 0$ such that

$$|a^*(u_h, \Pi_h^* w_h)| \leq M\|u_h\|_1\|w_h\|_1, \quad \forall u_h, w_h \in U_h. \tag{2.12}$$

Lemma 2.5 The bilinear form $a^*(\cdot, \Pi_h^*\cdot) : U_h \times U_h \to R$ is positive definite, that is, there exists a positive constant α such that, for h sufficiently small

$$a^*(u_h, \Pi_h^* u_h) \geq \alpha\|u_h\|_1^2, \quad \forall u_h \in U_h. \tag{2.13}$$

We now present a very useful lemma with respect to the bilinear form $a(\cdot,\cdot)$. For simplicity, we write

$$d(u - u_h, w_h) = a(u - u_h, w_h) + a^*(u - u_h, \Pi_h^* w_h), \quad \forall u_h, w_h \in U_h. \tag{2.14}$$

Lemma 2.6 [4] Let $u \in W^{3,q}(I)$, then, for $u_h, w_h \in U_h$,

$$|d(u - u_h, w_h)| \leq Ch[|u - u_h|_{1,p}|w_h|_{1,p'} + h|u|_{3,q}|w_h|_{1,q'}], \tag{2.15}$$

with $1 \leq p,\ q \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

We prove another useful lemma similar to Lemma 2.6.

Lemma 2.7 Let $u \in W^{2,q}(I)$, then, for $u_h, w_h \in U_h$,

$$|d(u - u_h, w_h)| \leq Ch[|u - u_h|_{1,p}|w_h|_{1,p'} + |u|_{2,q}|w_h|_{1,q'}], \tag{2.16}$$

with $1 \leq p,\ q \leq \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$.

Proof. By making use of (2.3), we have, for u_h, $w_h \in U_h$

$$\begin{aligned} a(u - u_h, w_h) &= \int_a^b p(u - u_h)' w_h' dx \\ &= \sum_{j=1}^n [\int_{x_{j-1}}^{x_j} (p - p_{j-\frac{1}{2}})(u - u_h)' w_h' dx \\ &\quad + \int_{x_{j-1}}^{x_j} p_{j-\frac{1}{2}}(u - u_h)' dx \frac{w_j - w_{j-1}}{h_j}] \\ &= \sum_{j=1}^n \int_{x_{j-1}}^{x_j} (p - p_{j-\frac{1}{2}})(u - u_h)' w_h' dx \\ &\quad + \sum_{j=1}^n p_{j-\frac{1}{2}}[u(x_j) - u(x_{j-1}) - u_j + u_{j-1}] dx \frac{w_j - w_{j-1}}{h_j}, \end{aligned}$$

where $u_j = u_h(x_j)$, $w_j = w_h(x_j)$.

On the other hand, by (2.4), (2.3) and noting $w_0 = w_n = 0$, we have,

$$\begin{aligned} a^*(u-u_h,\Pi_h^* w_h) &= \sum_{j=1}^{n-1} w_j a^*(u-u_h,\psi_j) \\ &= \sum_{j=1}^{n-1}[p_{j-\frac{1}{2}}(u-u_h)'_{j-\frac{1}{2}} - p_{j+\frac{1}{2}}(u-u_h)'_{j+\frac{1}{2}}]w_j \\ &= \sum_{j=1}^{n-1} p_{j-\frac{1}{2}}(u-u_h)'_{j-\frac{1}{2}} w_j - \sum_{j=2}^{n} p_{j-\frac{1}{2}}(u-u_h)'_{j-\frac{1}{2}} w_{j-1} \\ &= \sum_{j=1}^{n} p_{j-\frac{1}{2}}(u-u_h)'_{j-\frac{1}{2}}(w_j - w_{j-1}) \\ &= \sum_{j=1}^{n} p_{j-\frac{1}{2}}[h_j u'(x_{j-\frac{1}{2}}) - u_j + u_{j-1}]\frac{w_j - w_{j-1}}{h_j}. \end{aligned}$$

Substitute the above relations into (2.14) and express (2.14) by the following

$$d(u-u_h,w_h) = \sum_{j=1}^{2} E_i(u-u_h,w_h),$$

where

$$\begin{aligned} E_1(u-u_h,w_h) &= \sum_{j=1}^{n}\int_{x_{j-1}}^{x_j}(p-p_{j-\frac{1}{2}})(u-u_h)'w_h' dx, \\ E_2(u-u_h,w_h) &= \sum_{j=1}^{n} p_{j-\frac{1}{2}}[u(x_j) - u(x_{j-1}) - h_j u'(x_{j-\frac{1}{2}})]\frac{w_j - w_{j-1}}{h_j}. \end{aligned}$$

Thus

$$|E_1(u-u_h,w_h)| \le Ch|u-u_h|_{1,p}|w_h|_{1,p'}.$$

Applying Taylor's formula with integral type remainder

$$\begin{aligned} g(x_j) - g(x_{j-1}) &= g'(x_{j-\frac{1}{2}})h_j + \int_{x_{j-\frac{1}{2}}}^{x_j} g''(x)(x_j - x)dx \\ &\quad - \int_{x_{j-\frac{1}{2}}}^{x_{j-1}} g''(x)(x_{j-1} - x)dx, \end{aligned}$$

we have

$$|E_2(u-u_h,w_h)| \le Ch\int_a^b |u''w_h'|dx \le Ch|u|_{2,q}|w_h|_{1,q'}.$$

Hence the conclusion (2.16) is a consequence of combination of estimates for E_1 and E_2. This completes the proof.

In order to derive maximum norm error estimates, we need to define the Green functions associated with the bilinear form $a(\cdot,\cdot)$ (See [9]). Let $G_z^h \in U_h$ and $G_z^* \in H_0^1(I)$ be the discrete Green function and pre-Green function respectively, $\partial_z G_z^*$ the directional derivative of G_z^* along some direction with respect to z. Then G_z^h and $\partial_z G_z^h$ are the finite element approximations to G_z^* and $\partial_z G_z^*$, respectively. From [9], we know that

$$\|G_z^*\|_{2,1} + \|\partial_z G_z^h\|_{1,1} \le C. \tag{2.17}$$

Lemma 2.8 The discrete Green's function G_z^h possess the following property.

$$\|G_z^h\|_{1,\infty} \le C. \tag{2.18}$$

Proof. By using of inverse property, (2.5) and noting that $W^{2,1}(I) \hookrightarrow w^{1,\infty}(I)$, we obtain

$$\begin{aligned} \|G_z^h\|_{1,\infty} &\le \|G_z^*\|_{1,\infty} + \|G_z^* - \Pi_h G_z^*\|_{1,\infty} + \|\Pi_h G_z^* - G_z^h\|_{1,\infty} \\ &\le C\|G_z^*\|_{1,\infty} + Ch^{-\frac{1}{2}}\|\Pi_h G_z^* - G_z^h\|_1 \\ &\le C\|G_z^*\|_{1,\infty} + Ch^{-\frac{1}{2}}(\|\Pi_h G_z^* - G_z^*\|_1 + \|G_z^* - G_z^h\|_1) \\ &\le C\|G_z^*\|_{2,1} + Ch^{\frac{1}{2}}\|G_z^*\|_2. \end{aligned} \tag{2.19}$$

Let $\delta_z^h \in U_h$ be the discrete Delta function defined by

$$(v, \delta_z^h) = v(z), \quad \forall v \in U_h,$$

Then the above defination and the inverse property imply that

$$\begin{aligned} \|\delta_z^h\|_0^2 &= (\delta_z^h, \delta_z^h) &= |\delta_z^h(z)| \\ &\le \|\delta_z^h\|_{1,\infty} &\le Ch^{-\frac{1}{2}}\|\delta_z^h\|_0, \end{aligned}$$

So also from [9], we have

$$\|G_z^*\|_2 \le C\|\delta_z^h\|_0 \le Ch^{-\frac{1}{2}},$$

which together with (2.17) and (2.19) completes the proof.

3 Two-boundary Value problem

In this section, we consider the problem (1.1).

The first order generalized difference scheme for problem (1.1) is to find $u_h \in U_h$ such that

$$a^*(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{3.1}$$

Using the defination of the operator Π_h^* in section 2, we can obtain that the generalized difference scheme (3.1) is equivalent to finding $u_h \in U_h$, such that

$$a^*(u_h, \Pi_h^* w_h) = (f, \Pi_h^* w_h), \quad \forall w_h \in U_h. \tag{3.2}$$

Combining the results of Lax-Milgram's Lemma, Lemmas 2.4 and 2.5, we have the solvability theorem.

Theorem 3.1 The first order generalized difference scheme (3.1) has exactly one solution for h sufficiently small.

In this section, we denote $u \in H_0^1(I)$ the weak solution of problem (1.1) and $u_h \in U_h$ the solution of (3.1).

In view of the generalized Galerkin variational principle (See [7]), we have

$$a^*(u, v_h) = (f, v_h), \qquad \forall v_h \in V_h.$$

So that, by (3.1),

$$a^*(u - u_h, v_h) = 0, \qquad \forall v_h \in V_h. \tag{3.3}$$

Now, we show the error estimate of $u - u_h$ in $W^{1,p}(I)$ $(2 \le p \le \infty)$.

Theorem 3.2 If $u \in W^{2,p}(I)$ $(2 \le p < \infty)$ and h sufficiently small, then the following error estimate holds:

$$\|u - u_h\|_{1,p} \le Ch\|u\|_{2,p}, \quad 2 \le p < \infty. \tag{3.4}$$

Proof. We first introduce an auxiliary problem. Denote ϕ_x to be the derivative of ϕ and let $\Phi \in H_0^1(I)$ be the solution of

$$a(v, \Phi) = -(v, \phi_x), \quad \forall v \in H_0^1(I), \tag{3.5}$$

and there is a priori estimate

$$\|\Phi\|_{1,p'} \le C\|\phi\|_{0,p'}, \quad p' = \frac{p}{p-1}. \tag{3.6}$$

Let $\tilde{\Phi}$ and $\tilde{u}$ denote the standard finite element solutions of the problem (3.5) and (1.1) respectively. Then we have[2]

$$\begin{aligned}
a(v_h, \Phi - \tilde{\Phi}) &= 0, && \forall v_h \in U_h, && (3.7)\\
a(u - \tilde{u}, v_h) &= 0, && \forall v_h \in U_h, && (3.8)\\
\|u - \tilde{u}\|_{s,q} &\le Ch^{2-s}\|u\|_{2,q}, && s = 0, 1, \quad 2 \le q \le \infty. && (3.9)
\end{aligned}$$

By virtue of Green formula, (3.5), (3.7), (3.8), (3.3), Lemma 2.7, (3.9) and (3.6), we obtain that

$$\begin{aligned}((u-u_h)_x,\phi) &= -(u-u_h,\phi_x)\\ &= a(u-u_h,\Phi)\\ &= a(u-u_h,\Phi-\tilde{\Phi})+a(u-u_h,\tilde{\Phi})\\ &= a(u-\tilde{u},\Phi)+d(u-u_h,\tilde{\Phi})\\ &\le C[\|u-\tilde{u}\|_{1,p}\|\Phi\|_{1,p'}+h(|u-u_h|_{1,p}+|u|_{2,p})|\tilde{\Phi}|_{1,p'}]\\ &\le Ch(\|u-u_h\|_{1,p}+\|u\|_{2,p})\|\Phi\|_{1,p'}\\ &\le Ch(\|u-u_h\|_{1,p}+\|u\|_{2,p})\|\phi\|_{0,p'}.\end{aligned}$$

Thus

$$\|(u-u_h)_x\|_{0,p}=\sup_{\phi\in L_{p'}(I)}\frac{((u-u_h)_x,\phi)}{\|\phi\|_{0,p'}}\le Ch(\|u-u_h\|_{1,p}+\|u\|_{2,p}).$$

Therefore

$$\|u-u_h\|_{1,p}\le Ch(\|u-u_h\|_{1,p}+\|u\|_{2,p}), \tag{3.10}$$

where we have used the equivalence of the norms $\|\cdot\|_{1,p}$ and $|\cdot|_{1,p}$. Let h sufficiently small, such that $Ch\le\frac{1}{2}$, then the theorem follows at once from (3.10).

Theorem 3.3 If $u\in W^{2,\infty}(I)$ and h sufficiently small, then we have the following error estimate:

$$\|u-u_h\|_{1,\infty}\le Ch\|u\|_{2,\infty}. \tag{3.11}$$

Proof. The definition of $\partial_z G_z^h$, (3.8), (3.3), Lemma 2.7, (3.9), and (2.17) imply that

$$\begin{aligned}\partial_z(\tilde{u}-u_h)(z) &= a(\tilde{u}-u_h,\partial_z G_z^h)\\ &= a(u-u_h,\partial_z G_z^h)\\ &= d(u-u_h,\partial_z G_z^h)\\ &\le Ch(|u-u_h|_{1,\infty}+|u|_{2,\infty})|\partial_z G_z^h|_{1,1}\\ &\le Ch(|u-\tilde{u}|_{1,\infty}+|\tilde{u}-u_h|_{1,\infty}+|u|_{2,\infty})\\ &\le C(h^2\|u\|_{2,\infty}+h|\tilde{u}-u_h|_{1,\infty}).\end{aligned} \tag{3.12}$$

Hence

$$\|\tilde{u}-u_h\|_{1,\infty}\le C(h|\tilde{u}-u_h|_{1,\infty}+h^2\|u\|_{2,\infty}). \tag{3.13}$$

By letting h sufficiently small in the above inequality and using the triangle inequality

$$\|u-u_h\|_{1,\infty}\le\|u-\tilde{u}\|_{1,\infty}+\|\tilde{u}-u_h\|_{1,\infty},$$

we complete the proof also from (3.9).

Combining Theorems 3.2 and 3.3, we immediately derive the following.

Theorem 3.4 If $u \in W^{2,p}(I)$ $(2 \le p \le \infty)$ and h sufficiently small, then we have the following error estimate:

$$\|u - u_h\|_{1,p} \le Ch\|u\|_{2,p}, \quad 2 \le p \le \infty. \tag{3.14}$$

We then demonstrate the estimates of $u - u_h$ in $L_p(I)$ $(2 \le p \le \infty)$. From [4] and [5], we know that

$$\begin{aligned} \|u - u_h\|_0 &\le Ch^2\|u\|_{3,1}, \\ \|u - u_h\|_{0,\infty} &\le Ch^2\|u\|_3. \end{aligned}$$

The result is not perfect. We will modify the proof of the case of $p = \infty$ to reduce the demand of the smoothness of the function u. Then the case of $2 \le p < \infty$ is the straight result of the case of $p = \infty$.

Theorem 3.5 If $u \in W^{3,1}(I)$ and h sufficiently small, the following error estimate holds:

$$\|u - u_h\|_{0,\infty} \le Ch^2\|u\|_{3,1}. \tag{3.15}$$

Proof. Noting that the definition of G_z^h, (3.12), Lemma 2.6, (3.14), (2.18) and Sobolev imbedding inequality, we deduce that

$$\begin{aligned} (\tilde{u} - u_h)(z) &= a(\tilde{u} - u_h, G_z^h) \\ &= d(u - u_h, G_z^h) \\ &\le Ch(|u - u_h|_1 + h|u|_{3.1})|G_z^h|_{1,\infty} \\ &\le Ch^2(|u|_2 + |u|_{3.1}) \\ &\le Ch^2\|u\|_{3.1}, \end{aligned}$$

which together with (3.9), Sobolev imbedding inequality and triangle inequality completes the proof.

An application of the above theorem and the inequality

$$\|u - u_h\|_{0,p} \le C\|u - u_h\|_{0,\infty}, \qquad 2 \le p < \infty.$$

immediately yields the following result.

Corollary 3.1 Under the hypotheses of Theorem 3.5, we have the following error estimate:

$$\|u - u_h\|_{0,p} \le Ch^2\|u\|_{3,1}, \qquad 2 \le p \le \infty. \tag{3.16}$$

We now turn to the superconvergence estimate of $\tilde{u} - u_h$ in $W^{1,p}(I)$ $(2 \le p \le \infty)$.

Theorem 3.6 If $u \in W^{3,p}(I)$ $(2 \le p \le \infty)$ and h sufficiently small, then we have

$$\|\tilde{u} - u_h\|_{1,p} \le Ch^2\|u\|_{3,p}, \qquad 2 \le p \le \infty. \tag{3.17}$$

Proof. We first consider the case of $2 \le p < \infty$.

By Lemma 2.6, Theorem 3.2, and a similar analysis to that in Theorem 3.2, we obtain that

$$\begin{aligned} ((\tilde{u}-u_h)_x, \phi) &= -(\tilde{u}-u_h, \phi_x) \\ &= a(\tilde{u}-u_h, \Phi) \\ &= a(\tilde{u}-u_h, \tilde{\Phi}) \\ &= d(u-u_h, \tilde{\Phi}) \\ &\le Ch(|u-u_h|_{1,p} + h|u|_{3,p})|\Phi|_{1,p'} \\ &\le Ch^2(\|u\|_{2,p} + |u|_{3,p})\|\Phi\|_{1,p'}. \end{aligned}$$

accordingly, (3.17) is derived from the above inequality and (3.6), for $2 \le p < \infty$.

As far as the case of $p = \infty$ is concerned, it suffices to see the proof of Theorem 3.3 and the difference is that we use Lemma 2.6 instead of Lemma 2.7.

4 One-Dimensional Parabolic Problem

In this section, we consider the problem (1.2) on the base of the results derived in section 3.

Define the semi-discrete generalized difference scheme for problem (1.2): Find $u_h(t)$: $[0,T] \to U_h$ such that

$$\begin{cases} (a) & (u_{h,t}, v_h) + a^*(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \quad 0 < t \le T, \\ (a) & u_h(0) = u_{0,h}, \end{cases} \tag{4.1}$$

where $u_{h,t} = \frac{\partial u_h}{\partial t}$, $u_{0,h} \in U_h$ is taken the generalized elliptic projection $R_h^* u_0$ of u_0 defined in (4.2) below.

It can been proved that (4.1) has a unique solution for any $f \in L_2(I)$ (See [4], [7]).

For later use, we introduce the generalized elliptic projection operator $R_h^* : H^2(I) \cap H_0^1(I) \to U_h$ defined by

$$a^*(R_h^* w, v_h) = a^*(w, v_h), \qquad \forall v_h \in V_h. \tag{4.2}$$

It is easily seen that, from Lemmas 2.4 and 2.5, $R_h^* w$ is uniquely determined by (4.2) for any given $w \in H^1(I) \cap H_0^1(I)$.

Applying Theorems 3.4 and 3.6, we have the following error estimates.

Lemma 4.1 Let R_h^* be defined by (4.2), then

$$\begin{aligned} (a) \quad & \|w - R_h^* w\|_{1,p} \le Ch\|w\|_{2,p}, \quad 2 \le p \le \infty; \\ (b) \quad & \|w - R_h^* w\|_{0,p} \le Ch^2\|w\|_{3,1}, \quad 2 \le p \le \infty. \end{aligned} \tag{4.3}$$

Throughout this section, we denote $u \in H_0^1(I)$ the weak solution of problem (1.2), $u_h \in U_h$ the solution of problem (4.1) and write

$$u - u_h = (u - R_h^* u) + (R_h^* u - u_h) = \eta + \xi.$$

Let $\tilde{u}$ be the finite element solution of problem (1.2), that is, $\tilde{u} \in U_h$ satisfy[8]

$$\begin{cases} (\tilde{u}_t, v_h) + a(\tilde{u}, v_h) &= (f, v_h), \qquad \forall v_h \in U_h, \\ \tilde{u}(0) &= \tilde{u}_0, \end{cases}$$

where $\tilde{u}_t = \frac{\partial \tilde{u}}{\partial t}$, $\tilde{u}_0 \in U_h$ is taken the elliptic projection $R_h u_0$ of u_0 defined as follows: Find $R_h u \in U_h$ such that

$$a(u - R_h u, v_h) = 0, \qquad \forall v_h \in U_h. \tag{4.4}$$

First, we give the superconvergence estimate of ξ in $W^{1,p}(I)$ $(2 \leq p \leq \infty)$. In order to get that we need to deduce the error estimate of ξ_t in $L_2(I)$.

Lemma 4.2 If $u_t(0) \in W^{3,1}(I)$, $u_{tt} \in L_2(0,t;w^{3,1}(I))$, then we have the following error estimate:

$$\|\xi_t\|_0 \leq Ch^2\{\|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}. \tag{4.5}$$

proof. Mutiplying (1.2) by v_h and integrating by parts, we have

$$(u_t, v_h) + a^*(u, v_h) = (f, v_h), \quad \forall v_h \in V_h. \tag{4.6}$$

Substracting (4.6) from (4.1) and applying (4.2), we obtain

$$(\xi_t, v_h) + a^*(\xi, v_h) = -(\eta_t, v_h), \quad \forall v_h \in V_h. \tag{4.7}$$

Taking $t = 0$ in (4.7) and noting that $u_h(0) = R_h^* u_h$ implies $\xi(0) = 0$, then taking $v_h(0) = \Pi_h^* \xi_t(0)$, we have, also by Lemma 2.2 and ε-inequality, that

$$\begin{aligned} |||\xi_t(0)|||_0^2 &= -(\eta_t(0), \Pi_h^* \xi_t(0)) \\ &\leq C\|\eta_t(0)\|_0 \|\Pi_h^* \xi_t(0)\|_0 \\ &\leq C\|\eta_t(0)\|_0^2 + \tfrac{1}{2}|||\xi_t(0)|||_0^2, \end{aligned}$$

where $|||\cdot|||$ is defined in Lemma 2.2. Then by (4.3b)

$$|||\xi_t(0)|||_0 \leq C\|\eta_t(0)\|_0 \leq Ch^2\|u_t(0)\|_{3,1}. \tag{4.8}$$

Differentiate (4.7) with respect to t and take $v_h = \Pi_h^* \xi_t$ to get

$$(\xi_{tt}, \Pi_h^* \xi_t) + a^*(\xi_t, \Pi_h^* \xi_t) = -(\eta_{tt}, \Pi_h^* \xi_t). \tag{4.9}$$

Lemmas 2.3, 2.5 and 2.2 now imply that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}(\xi_t, \Pi_h^* \xi_t) + C_*\|\xi_t\|_1^2 &\leq (\xi_{tt}, \Pi_h^* \xi_t) + a^*(\xi_t, \Pi_h^* \xi_t) \\ &\leq C(\|\eta_{tt}\|_0^2 + |||\xi_t|||_0^2). \end{aligned} \tag{4.10}$$

Integrating (4.10) with respect to t, we have

$$|||\xi_t|||_0^2 + \int_0^t \|\xi_t\|_1^2 d\tau \le C(|||\xi_t(0)|||_0^2 + \int_0^t \|\eta_{tt}\|_0^2 d\tau + \int_0^t |||\xi_t|||_0^2 d\tau).$$

By Gronwall Lemma, (4.8) and (4.3b), we have

$$\begin{aligned} |||\xi_t|||_0^2 &\le C(|||\xi_t(0)|||_0^2 + \int_0^t \|\eta_{tt}\|_0^2 d\tau) \\ &\le Ch^4(\|u_t(0)\|_{3,1}^2 + \int_0^t \|u_{tt}\|_{3,1}^2 d\tau). \end{aligned} \tag{4.11}$$

(4.11) and Lemma 2.2 imply (4.5).

Theorem 4.1 If $u_t(0)$, $u_{tt} \in W^{3,1}(I)$, $u \in W^{3,p}(I)$ and h sufficiently small, then the following superconvergence estimate of ξ in $W^{1,p}(I)$ $(2 \le p \le \infty)$ holds:

$$\|\xi\|_{1,p} \le Ch^2\{\|u\|_{3,p} + \|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}, \quad 2 \le p \le \infty. \tag{4.12}$$

Proof. (i) Let us first consider the case of $2 \le p < \infty$.

Using Green formula, (3.5), (3.7), (4.2) and (4.7), we write

$$\begin{aligned} (\xi_x, \phi) &= -(\xi, \phi_x) = a(\xi, \Phi) = a(\xi, R_h\Phi) \\ &= a(R_h^* u - u, R_h\Phi) + a(u - u_h, R_h\Phi) \\ &= a(R_h^* u - u, R_h\Phi) - a^*(R_h^* u - u, \Pi_h^* R_h\Phi) \\ &\quad + a(u - u_h, R_h\Phi) - a^*(u - u_h, \Pi_h^* R_h\Phi) + a^*(\xi, \Pi_h^* R_h\Phi) \\ &= d(R_h^* u - u, R_h\Phi) + d(u - u_h, R_h\Phi) - (\eta_t + \xi_t, \Pi_h^* R_h\Phi) \\ &= E_1 + E_2 + E_3. \end{aligned} \tag{4.13}$$

Applying Lemma 2.6 and (4.3a), we get

$$\begin{aligned} E_1 &\le Ch(|R_h^* u - u|_{1,p} + h|u|_{3,p})|R_h\Phi|_{1,p'} \\ &\le Ch^2\|u\|_{3,p}|\Phi|_{1,p'}, \end{aligned}$$

and

$$\begin{aligned} E_2 &\le Ch(|u - u_h|_{1,p} + h|u|_{3,p})|R_h\Phi|_{1,p'} \\ &\le Ch(|u - R_h^* u|_{1,p} + |\xi|_{1,p} + h|u|_{3,p})|\Phi|_{1,p'} \\ &\le C(h|\xi|_{1,p} + h^2\|u\|_{3,p})\|\Phi\|_{1,p'}. \end{aligned}$$

Also we know, from (4.3b),that

$$\begin{aligned} \|\eta_t\|_0 &\le \|\eta_t(0)\|_0 + \int_0^t \|\eta_{tt}\|_0 d\tau \\ &\le Ch^2\{\|u_t(0)\|_{3,1} + \int_0^t \|u_{tt}\|_{3,1} d\tau\} \\ &\le Ch^2\{\|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}, \end{aligned}$$

which together with (4.5) and Sobolev imbedding inequality implies that

$$\begin{aligned} E_3 &\leq (\|\eta_t\|_0 + \|\xi_t\|_0)\|\Pi_h^* R_h \Phi\|_0 \\ &\leq Ch^2\{(\|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}\|\Phi\|_{1,p'}. \end{aligned}$$

Combining the estimates of $E_1 - E_3$, we obtain also by (3.6) and (4.13) that

$$\begin{aligned} \|\xi\|_{1,p} &\leq C \sup_{\phi \in L_{p'}(I)} \frac{(\xi_x, \phi)}{\|\phi\|_{0,p'}} \\ &\leq C\{h\|\xi\|_{1,p} + h^2[\|u\|_{3,p} + \|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}]\}. \end{aligned}$$

By letting h sufficiently small such that $Ch \leq \frac{1}{2}$, we can complete the proof of the case of $2 \leq p < \infty$.

(ii) Let us next consider the case of $p = \infty$. By virtue of the definition of $\partial_z G_z^h$, we have

$$\partial_z \xi(z) = a(\xi, \partial_z G_z^h),$$

consequently, upon replacing $R_h \Phi$ by $\partial_z G_z^h$, p by ∞, p' by 1 in part (i), likewise, we obtain the conclusion.

Arguing as in the proof of Lemma 4.2, we find that if we use (4.3a) instead of (4.3b) in (4.8) and (4.11), we can get the following lemma.

Lemma 4.3 If $u_t(0) \in W^{2,p}(I)$, $u_{tt} \in L_2(0, t; W^{2,p}(I))$, then the following error estimate holds:

$$\|\xi_t\|_0 \leq Ch\{\|u_t(0)\|_{2,p} + (\int_0^t \|u_{tt}\|_{2,p}^2 d\tau)^{\frac{1}{2}}\}, \quad 2 \leq p \leq \infty. \tag{4.14}$$

As to the error estimate of $u - u_h$ in $W^{1,p}(I)$, we obtain it by the similar way to getting Theorem 4.1. In the proof of Theorem 4.1, we use Lemma 2.7 instead of Lemma 2.6 and (4.14) instead of (4.5) for the error estimates of $E_1 - E_3$ to derive

$$\|\xi\|_{1,p} \leq Ch\{\|u_t(0)\|_{2,p} + \|u\|_{2,p} + (\int_0^t \|u_{tt}\|_{2,p}^2 d\tau)^{\frac{1}{2}}\}, \quad 2 \leq p \leq \infty, \tag{4.15}$$

which together with (4.3a), by using triangle inequality leads to the following theorem.

Theorem 4.2 If $u_t(0)$, $u \in W^{2,p}(I)$, $u_{tt} \in L_2(0, t; w^{2,p}(I))$ and h sufficiently small, then we have the following error estimate, for $2 \leq p \leq \infty$

$$\|u - u_h\|_{1,p} \leq Ch\{\|u_t(0)\|_{2,p} + \|u\|_{2,p} + (\int_0^t \|u_{tt}\|_{2,p}^2 d\tau)^{\frac{1}{2}}\}. \tag{4.16}$$

Now we turn to the error estimates of $u - u_h$ in $L_p(I)$ $(2 \le p \le \infty)$. We only demonstrate the case of $p = \infty$ and the case of $2 \le p < \infty$ is a immediate result.

Theorem 4.3 If $u_t(0)$, $u \in W^{3,1}(I)$, $u_{tt} \in L_2(0,t;w^{3,1}(I))$ and h sufficiently small, then the following error estimate holds:

$$\|u-u_h\|_{0,\infty} \le Ch^2\{\|u_t(0)\|_{3,1} + \|u\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}. \tag{4.17}$$

Proof. In view of the definition of G_z^h and (4.13), we obtain that

$$\begin{aligned} \xi(z) &= a(\xi, G_z^h) \\ &= d(R_h^* u - u, G_z^h) + d(u-u_h, G_z^h) - (\eta_t + \xi_t, \Pi_h^* G_z^h) \\ &= Q_1 + Q_2 + Q_3. \end{aligned}$$

Lemma 2.6, (4.3a), (2.18) and Sobolev imbedding inequality imply that

$$\begin{aligned} Q_1 &\le Ch(|R_h^* u - u|_1 + h|u|_{3,1})|G_z^h|_{1,\infty} \\ &\le Ch^2\|u\|_{3,1}, \end{aligned}$$

and that, also from (4.16)

$$\begin{aligned} Q_2 &\le Ch(|u-u_h|_1 + h|u|_{3,1})|G_z^h|_{1,\infty} \\ &\le Ch^2\{\|u_t(0)\|_2 + \|u\|_2 + (\int_0^t \|u_{tt}\|_2^2 d\tau)^{\frac{1}{2}} + |u|_{3,1}\} \\ &\le Ch^2\{\|u_t(0)\|_{3,1} + \|u\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}. \end{aligned}$$

Taking into account (4.3b), (4.5) and Sobolev imbedding inequality, we find that

$$\begin{aligned} Q_3 &\le (\|\eta_t\|_0 + \|\xi_t\|_0)|G_z^h|_{1,\infty} \\ &\le Ch^2\{\|u_t(0)\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}. \end{aligned}$$

The inequality (4.17) follows by combining the error estimates of $Q_1 - Q_3$, (4.3b) and triangle inequality.

Corollary 4.1 Under the hypotheses of Theorem 4.3, we have the following error estimate, for $2 \le p \le \infty$

$$\|u-u_h\|_{0,p} \le Ch^2\{\|u_t(0)\|_{3,1} + \|u\|_{3,1} + (\int_0^t \|u_{tt}\|_{3,1}^2 d\tau)^{\frac{1}{2}}\}. \tag{4.18}$$

Finally, we deduce the superconvergence estimates of $\tilde{u}-u_h$ and $R_h^*u-\tilde{u}$ in $W^{1,p}(I)$ $(2\leq p\leq\infty)$.

From [3], we know that

$$\|\tilde{u}-R_hu\|_{1,p}\leq Ch^2(\int_0^t\|u_t\|_{2,p}^2d\tau)^{\frac{1}{2}},\qquad 2\leq p\leq\infty. \tag{4.19}$$

Theorem 4.4 The following superconvergence estimate holds, for $2\leq p\leq\infty$

$$\begin{aligned}\|\tilde{u}-u_h\|_{1,p} \leq\ & Ch^2\{\|u(0)\|_{3,p}+\|u_t(0)\|_{3,1}\\ &+(\int_0^t\|u_t\|_{3,p}^2d\tau)^{\frac{1}{2}}+(\int_0^t\|u_{tt}\|_{3,1}^2d\tau)^{\frac{1}{2}}\}.\end{aligned} \tag{4.20}$$

Proof. Using (3.5) and (4.4), we have

$$\begin{aligned}((\tilde{u}-u_h)_x,\phi) &= -(\tilde{u}-u_h,\phi_x)\\ &= a(\tilde{u}-u_h,\Phi)\ =\ a(\tilde{u}-u_h,R_h\Phi)\\ &= a(\tilde{u}-R_hu,R_h\Phi)+a(u-u_h,R_h\Phi)\\ &\leq C\|\tilde{u}-R_hu\|_{1,p}\|R_h\Phi\|_{1,p'}\\ &\quad +d(u-u_h,R_h\Phi)+a^*(u-u_h,R_h\Phi).\end{aligned}$$

Therefore, similar to Theorem 4.1, the proof is easily complete also by (4.19).

Theorems 4.1 and 4.4 together with

$$\|R_h^*u-\tilde{u}\|_{1,p}\leq\|\xi\|_{1,p}+\|\tilde{u}-u_h\|_{1,p},$$

yield the following

Corollary 4.2 The following superconvergence estimate holds, for $2\leq p\leq\infty$

$$\begin{aligned}\|R_h^*u-\tilde{u}\|_{1,p} \leq\ & Ch^2\{\|u(0)\|_{3,p}+\|u_t(0)\|_{3,1}\\ &+(\int_0^t\|u_t\|_{3,p}^2d\tau)^{\frac{1}{2}}+(\int_0^t\|u_{tt}\|_{3,1}^2d\tau)^{\frac{1}{2}}\}.\end{aligned} \tag{4.21}$$

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