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Model-Free Interval Prediction in a Class of Time Series with Varying Coefficients

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Abstract

Interval prediction based on the empirical distribution function for the class of time series with time varying coefficients is discussed. To this end, strong mixing property of the model is shown and results due to Fotopoulos et. al.(1994) are employed. A simulation study is presented to assess the accuracy of the proposed interval predictor.

Key Words and Phrases: Time Varying Coefficient, Strong Mixing, Sample Quantile Function, Model-Free Interval Prediction.

1. INTRODUCTION

Let $\{X_t,\ t = 0, \pm 1, \cdots\}$ be a general time varying coefficient process of order p satisfying the difference equation

$$X_t = \mathbf{\Phi_t}' X(t-1) + \epsilon_t, \tag{1.1}$$

where $\{\mathbf{\Phi_t}\} = \{(\phi_{t1}, \cdots, \phi_{tp})'\}$ is a arbitrary $(p \times 1)$ stationary vector process, $X(t-1) = (X_{t-1}, \cdots, X_{t-p})'$ is a $(p \times 1)$ vector of the recent past observations at time t and $\{\epsilon_t\}$ is a sequence of *i.i.d.* random errors with zero mean and variance σ_ϵ^2.

Model (1.1) is rich enough to cover the standard random coefficient process (Nicholls and Quinn, (1982)), Markov bilinear model (Tong(1981)) and their generalizations among others. When the error term ϵ_t is independent of $\mathbf{\Phi_t}$, the model can be viewed as the doubly stochastic model (Tjøstheim (1986)) and when the parameter $\mathbf{\Phi_t}$ is *i.i.d.* random vector (permitted) correlated with ϵ_t, the model is referred to as a generalized random coefficient autoregressive model (Hwang and Basawa(1998)). Thus, we are dealing with fairly general class of time varying coefficient models.

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The main goal of this paper is to derive the model-free one-step-ahead interval predictor of the general time varying coefficient models including generalized random coefficient process and doubly stochastic process, as special cases. It is shown that under some regularity conditions time varying coefficient processes under consideration satisfy the strong mixing condition. The prediction interval for a future observation can then be constructed adapting the lines as in Fotopoulos et al.(1994).

2. THE PRELIMINARY RESULTS

Consider the real vauled strictly stationary sequence of random variables $\{X_t; t = 0, \pm 1, \cdots\}$ with common marginal distribution $F(x)$. The empirical distribution function is denoted by $F_n(x)$, which is given by

$$F_n(x) = n^{-1} \sum_{t=1}^{n} I_{[X_t \leq x]}, \quad -\infty < x < \infty.$$

For fixed $0 < s < 1$, let $F_n^{-1}(s)$ denote the sample quantile function which is defined as

$$F_n^{-1}(s) = inf\{y : F_n(y) \geq s\}. \tag{2.1}$$

Regarding prediction, a model-free interval predictor is proposed by Butler(1981) for the independent case and Cho and Miller(1987) generalizes it to the dependent case. We refer to Fotopoulos et al.(1994) for the background and comprehensive treatment for the model-free interval prediction of the strictly stationary process with applications to ARMA processes. In those works the proposed class of the $100\alpha\%$ prediction interval is given by

$$\{I(\delta) = [F^{-1}(\delta), F^{-1}(\delta + \alpha)]; 0 \leq \delta \leq 1 - \alpha\}, \tag{2.2}$$

where δ mininizes the trimmed variance, i.e.,

$$\sigma^2(\delta) = \alpha^{-1} \int_{F^{-1}(\delta)}^{F^{-1}(\delta+\alpha)} x^2 dF(x) - \left\{ \alpha^{-1} \int_{F^{-1}(\delta)}^{F^{-1}(\delta+\alpha)} x dF(x) \right\}^2. \tag{2.3}$$

With unknown $F(\cdot)$, the derivation of the interval (2.2) may not be feasible. Denote the estimator of δ by $\hat{\delta}$ which is obtained from replacing $F(\cdot)$ by its sample version $F_n(\cdot)$. Substituting δ in (2.2) and (2.3) by $\hat{\delta}$, the estimated model free interval is then of the form

$$\hat{I}(\hat{\delta}) = [F_n^{-1}(\hat{\delta}), F_n^{-1}(\hat{\delta} + \alpha)]. \tag{2.4}$$

The following theorem is essential to construct the model-free prediction interval for the strictly stationary process with strong mixing coefficient $\gamma(k) = O(k^{-8})$.

Theorem 1(Fotopoulos et al., 1994) Let $\{X_t\}$ be a strictly stationary real valued sequence of random variables satisfying the strong mixing property with strong mixing coefficient such that $\gamma(k) = O(k^{-8})$. We then have as n goes to ∞ ,

$$\hat{\delta} \xrightarrow{p} \delta \tag{2.5}$$

and

$$P(F_n^{-1}(\hat{\delta}) \leq X_{n+1} \leq F_n^{-1}(\hat{\delta} + \alpha)) \longrightarrow \alpha. \tag{2.6}$$

Notice that the interval (2.4) becomes then a asymptotic $100\alpha\%$ prediction interval.

3. STRONG MIXING PROPERTY OF THE PROCESS

Due to Theorem 1 the model-free one-step-ahead prediction interval for the time varying coefficient process can be constructed via (2.4) by showing the strong mixing property of the model. To this end, m-dependence structure will be imposed on the random coefficient $\mathbf{\Phi_t}$.

Theorem 2 Let $\{X_t\}$ be the strictly stationary process with distribution function $F(x)$ from the difference equation ;

$$X_t = \mathbf{\Phi_t}' X(t-1) + \epsilon_t$$

where $\mathbf{\Phi_t} = (\phi_{t1}, \cdots, \phi_{tp})'$ is a m-dependent arbitrary $(p \times 1)$ vector process with probability $1 - \beta$, and zero vector with remaining probability $\beta > 0$, and $\{\epsilon_t\}$ is a sequence of *i.i.d.* random errors with arbitrary distribution.

Then $\{X_t\}$ is a strong mixing process with exponentially decreasing mixing coefficient, *i.e.*, $\gamma(k) = O(e^{-\tau k})$ for some $\tau > 0$.

Proof. Let c be the maximum among p: order of process and m: dependence order of parameter process Φ, *i.e.*, $c = max(p, m)$. Let $\sigma(\cdot)$ denote the σ-field generated by $\cdot$, and for $-\infty \leq m \leq n \leq \infty$, define $\mathcal{F}_m^n = \sigma(X_k; m \leq k \leq n)$. Choose A and B such that $A \in \mathcal{F}_0^n$ and $B \in \mathcal{F}_{n+k}^N$ where $k > c$.

Consider the *i.i.d* random variables $\{V_t\}$ taking values 0 and 1, which are independent of X_0 and ϵ_t, such that $P(V_t = 0) = \beta$ and $P(V_t = 1) = 1 - \beta$. It may be noted that $X_t = \epsilon_t$ on the event $V_t = 0$, on $V_t = 1$, $X_t = \mathbf{\Phi_t}' X(t-1) + \epsilon_t$.

One can write $k - 1 = lc + q$ where $l \geq 1$ and $0 \leq q \leq c - 1$, and let π be the event on which there are c consecutive V_t such that $V_t = 0$ where $n + 1 \leq t \leq n + lc$. Clearly A and B are independent on the event π.

Following basically the similar lines as in Lee(1995), let π_i be the event such that $V_{n+1} = \cdots = V_{n+i-1} = 1$ and $V_{n+i} = \cdots = V_{n+i+c-1} = 0$ where $1 \leq i \leq lc - c + 1$, then on π_i

$$\{X_o, \cdots, X_{n+i-1}\} \perp \{X_{n+i+c}, \cdots, X_N\}$$

where $\perp$ stands for "independence" and π_i's are disjoint with $\pi = \cup_{i=1}^{lc-c+1} \pi_i$.

Then by the independence of $\{V_t\}$ and $\{\epsilon_t\}$, we have that

$$|P(A \cap B|\pi_i) - P(A)P(B|\pi_i)| = 0.$$

Notice that π_i are disjoint and they exhaust π. It then follows

$$\begin{aligned} |P(A \cap B|\pi) - P(A)P(B|\pi)| &\leq \sum_{i=1}^{lc-c+1} |P(A \cap B|\pi_i) - P(A)P(B|\pi_i)| \cdot P(\pi_i)/P(\pi) \\ &= 0. \end{aligned}$$

From the following fact :

$$|P(A \cap B|\pi^c) - P(A)P(B|\pi^c)| \leq 1,$$

it can be deduced that

$$\begin{aligned} |P(A \cap B) - P(A)P(B)| &\leq P(\pi^c)|P(A \cap B|\pi^c) - P(A)P(B|\pi^c)| \\ &\leq P(\pi^c). \end{aligned} \tag{3.1}$$

Also, an adaptation of the arguments in Lee(1995) leads to

$$|P(\pi^c)| = O((1-\beta^c)^{k/c}), \tag{3.2}$$

which essentially completes the proof. □

4. SIMULATION RESULTS

We consider simulated process of a general time varying coefficient process of order 1 where the parameter process is MA(1) with probability $1-\beta$ and zero with probability β. The MA(1) parameter process is generated from standard normal distribution. More precisely, the simulated process is of the form

$$X_t = \Phi_t X_{t-1} + \epsilon_t.$$

where

$$\Phi_t = \begin{cases} \theta\eta_{t-1} + \eta_t & w.p. \;\; 1-\beta \\ 0 & w.p. \;\; \beta > 0. \end{cases} \tag{4.1}$$

with $\{\eta_t\}$ is *i.i.d.* $N(0,1)$, and $\{\epsilon_t\}$ is *i.i.d.* random errors with distributions which will be specified later.

To see how the model-free prediction interval for one-step-ahead new observations works, we perform a simulation study by generating observations from (4.1) when the random errors follow : $N(0,1)$, Laplace(0,1), Uniform(-1,1) and Cauchy(0,1), respectively.

We generate the 151 observations using the SAS/IML, and discard the first 50 observations to minimize the effect of starting values. Based on the first remaining 100 observations, 90 % prediction interval via (2.4) is constructed, the last 101th observation is to be checked for coverage probability. The coverage percent obtained from 1000 replications is reported in Table 1.

It is observed that prediction interval given by (2.4) works well in the sense that most of the coverage percentages ranged from 89 to 91 %, which are similar to the results reported in Fotopoulos et al.(1994) for ARMA model. It is also noticed that the coverage percentage seems to remain unchanged from $\beta = 0.1$ to $\beta = 0.9$ for each fixed θ.

Accordingly, it is expected that the prediction interval as in (2.4) may well be employed for the case where Φ_t follows general linear processes with $\beta = 0$. Regarding this guess, the rigorous proof of the strong mixing property of the model with Φ_t being arbitrary will be left open for the future study.

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Table 1 Simulated coverage probability: **(i)** Error Process : Normal(0,1)

$\beta\backslash\theta$	-0.8	-0.5	-0.2	0.2	0.5	0.8
0.1	0.9018	0.8990	0.9002	0.9020	0.8986	0.8910
0.3	0.9068	0.9056	0.9064	0.8950	0.9004	0.8922
0.5	0.9056	0.9040	0.8938	0.8992	0.9024	0.9066
0.7	0.9064	0.9078	0.8978	0.8992	0.8998	0.9046
0.9	0.9048	0.9016	0.8992	0.9050	0.9036	0.9070

(ii) Error Process : Laplace(0,1)

$\beta\backslash\theta$	-0.8	-0.5	-0.2	0.2	0.5	0.8
0.1	0.9018	0.8952	0.9036	0.9000	0.9008	0.8974
0.3	0.9022	0.8964	0.9022	0.9096	0.8990	0.8934
0.5	0.9012	0.9036	0.8966	0.8964	0.8950	0.8940
0.7	0.8936	0.9026	0.8958	0.9006	0.9114	0.9012
0.9	0.9002	0.9066	0.8956	0.8992	0.8980	0.9030

(iii) Error Process : Uniform(-1,1)

$\beta\backslash\theta$	-0.8	-0.5	-0.2	0.2	0.5	0.8
0.1	0.8892	0.8988	0.8982	0.8998	0.8938	0.8972
0.3	0.9054	0.8988	0.9038	0.9046	0.9008	0.9016
0.5	0.9054	0.8962	0.9004	0.9018	0.8954	0.8926
0.7	0.9024	0.8986	0.8962	0.8976	0.9008	0.8950
0.9	0.8970	0.8992	0.8982	0.8950	0.9088	0.9028

(iv) Error Process : Cauchy(0,1)

$\beta\backslash\theta$	-0.8	-0.5	-0.2	0.2	0.5	0.8
0.1	0.8928	0.9012	0.9036	0.9020	0.9026	0.9024
0.3	0.8928	0.9042	0.9038	0.9088	0.8956	0.9036
0.5	0.9038	0.9014	0.9020	0.9074	0.9036	0.9046
0.7	0.8980	0.8992	0.9030	0.8962	0.9004	0.9018
0.9	0.8994	0.9086	0.9008	0.9016	0.9036	0.9082