Journal of the Korean
Data & Information Science Society
2000, Vol. 11, No.1, pp. 127 ~ 137

Hierarchical Bayesian Analysis for Stress-Strength Model in Normal Case

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Abstract

In this paper, we consider hierarchical Bayesian analysis for $P(Y < X)$ using Gibbs sampler, where X and Y are independent normal distributions with unknown means and variances, respectively. Also numerical study using real data is provided.

Key Words and Phrases: Gibbs sampler, Hierarchical Bayes, Reliability, Stress-strength model.

1. Introduction

The stress-strength model has been widely in a variety of areas including estimating for the reliability of a design procedures. This model was first introduced in 1950's and used on various applications in civil, aerospace engineering, et.al..

In the simplest stress-strength model, X is the strength placed on the unit by the operating environment and Y is the stress of the unit. A unit is able to perform its intended function if its strength is greater than the stress imposed upon it. In this paper, we define reliability (R) as probability that the unit performs its task satisfactorily. That is, reliability is the probability that the unit is strong enough to overcome the stress, that is, $R = P(Y < X)$. In particular, Related problems have been widely presented in the literature when X and Y have independent normal distributions, respectively.

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Let the strength distribution be normal with mean μ_1 and variance σ_1^2, and let the stress distribution be normal with mean μ_2 and variance σ_2^2 . Then the reliability of stress-strength model becomes

$$R = P(Y < X) = \Phi(\theta), \tag{1}$$

where $\theta = \frac{\mu_1 - \mu_2}{\sqrt{\sigma_1^2 + \sigma_2^2}}$.

For frequentist approach, first, Church and Hariss(1970) obtained an estimator and an approximate confidence interval for the reliability when the distribution of stress is known. Downton (1973) derived the uniformly minimum variance unbiased estimator of R. Reiser and Guttman (1986) considered statistical inference for R, where X and Y are independent normal variates with unknown means and variances, respectively. For the stress-strength model with explanatory variables, Duncan(1986) gave some specific examples and Guttman, Johnson, Bhattacharyya and Reiser(1988) obtained confidence limits for R. For the random effect model, Aminzadeh(1991) derived approximate confidence interval based on the asymptotic normal distribution for the R. Weerahandi and Johnson(1992) considered testing for reliability in a stress-strength model. Cho(1995) obtained some approximate confidence intervals for R when the stress and strength each depend on some explanatory variables, respectively.

In Bayesian approach, Enis and Geisser(1971) obtained predictive estimates and posterior Bayesian limits from a Bayesian viewpoint. Weerahandi and Johnson(1992) considered a Bayesian analysis for stress-strength model with conjugate prior. But study of hierarchical Bayes analysis for stress-strength model in normal case considered until now.

In this paper, we consider the hierarchical Bayes analysis for reliability of the stress-strength model in normal model using Gibbs sampler. In section 2, we review the Gibbs sampler. In section 3, we describe the computation methods for Bayes estimation for R using Gibbs sampler. In section 4, we implement the stress-strength model with an illustration from the rocket-motor experiment data.

2. Gibbs Sampler

For convenience, we define the following notations. Densities are denoted generically by brackets, so joint, conditional, and marginal forms, for example, appear as $[X, Y]$, $[X|Y]$, and $[X]$. Multiplication of densities is denoted by $*$; for example, $[X, Y] = [X|Y] * [Y]$. The process of marginalization (i.e., integration) is denoted by forms such as $[X|Y] = \int [X|Y, Z, W] * [Z|W, Y] * [W|Y]$, with the convention that all variables appearing in the integrand but not in resulting density have been integated out. Thus the integration is with respect to Z and W. More generally, we use notation such as $\int h(Z,W) * [W]$ to denote, for given Z, the expectation of the function $h(Z,W)$ with respect to the marginal distribution for W.

The Gibbs sampler is an iterative Monte Carlo integration method, developed formally by Geman and Geman(1984) in the context of image restoration. Gelfand and Smith(1990) developed the Gibbs sampler for fairly general parametric settings. To summarize the method briefly, suppose we have a collection of p r.v.'s $U_1, \cdots, U_p$ whose full conditional distributions, denoted generally by $[U_s|U_r, r \neq s]$, $s = 1, 2, \cdots, p$ are available for sampling. Under mild conditions, these full conditional distributions uniquely determine the full joint distribution $[U_1, \cdots, U_p]$ and hence all the marginal distributions $[U_s]$, $s = 1, 2, \cdots, p$.

The Gibbs sampler generates from the conditional distributions as follows: Given an arbitrary starting set of values $U_1^{(0)}, \cdots, U_p^{(0)}$, we draw $U_1^{(1)}$ from $[U_1|U_2^{(0)}, \cdots, U_p^{(0)}]$, $U_2^{(1)}$ from $[U_2|U_1^{(1)}, U_3^{(0)} \cdots, U_p^{(0)}]$, and so on up to $U_p^{(1)}$ from $[U_p|U_1^{(1)}, \cdots, U_{p-1}^{(1)}]$ to complete one iteration of the scheme. After t such iterations we arrive at a joint sample $(U_1^{(t)}, \cdots, U_p^{(t)})$ from $[U_1, \cdots, U_p]$. Under mild conditions, Geman and Geman(1984) showed that $(U_1^{(t)}, \cdots, U_p^{(t)}) \longrightarrow^d (U_1, \cdots, U_p) \sim [U_1, \cdots, U_p]$ as $t \longrightarrow \infty$. Hence for sufficiently large t, $U_s^{(t)}$ can be regarded as a sample from $[U_s]$.

Parallel replications l times yields l i.i.d. p-tuples: $(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, $j = 1, 2, \cdots, l$. For any function T of $U_1, \cdots, U_p$ whose expectation exists,

$$\frac{1}{l}\sum_{i=1}^{l} T(U_1^{(i)}, \cdots, U_p^{(i)}) \longrightarrow E[T(U_1, \cdots, U_p)], \ \textit{as } l \longrightarrow \infty \tag{2}$$

almost surely. The distribution of $(U_1, \cdots, U_p)$ can be approximated by the empirical distribution of $U_{1j}^{(t)}, \cdots, U_{pj}^{(t)}$, $j = 1, 2, \cdots, l$. Similarly the marginal of U_s can be approximated by the empirical distribution of $U_{sj}^{(t)}$. If $[U_s|U_r, r \neq s]$ can be computed, then

$$[\hat{U}_s] = \frac{1}{l}\sum_{j=1}^{l} [U_s|U_{rj}^{(t)}, \ r \neq s]. \tag{3}$$

For any $T(U_1, \cdots, U_p)$, let $T_j^{(t)} \equiv T(U_{1j}^{(t)}, \cdots, U_{pj}^{(t)})$, the empirical distribution of $T_1^{(t)}, \cdots, T_l^{(t)}$ provides an estimate of $[T(U_1, \cdots, U_p)]$.

3. Hierarchical Bayes Formulation for Stress-Strength Model

In this paper, we consider the general hierarchical Bayes model as following:

(i) Underline distributions of stress and strength are as followings, respectively.

$[x_i|\mu_1, \sigma_1^2] \sim^{i.i.d.} N(\mu_1, \sigma_1^2)$, where $i = 1, 2, \cdots, m$.

$[y_j|\mu_2, \sigma_2^2] \sim^{i.i.d.} N(\mu_2, \sigma_2^2)$, where $j = 1, 2, \cdots, n$.

(ii) Prior distributions of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ are given as followings, respectively.

$[\mu_1|c_1, d_1^2] \sim N(c_1, d_1^2)$,

$[\mu_2|c_2, d_2^2] \sim N(c_2, d_2^2)$,

$[\sigma_1^2|e_1, f_1] \sim IG(e_1, f_1)$,

$[\sigma_2^2|e_2, f_2] \sim IG(e_2, f_2)$,

where $IG(a, b)$ denotes an inverted gamma (a, b) random variable that has probability distribution function $f(z|a, b) = \frac{e^{-1/bx}}{\Gamma(a) \cdot b^a \cdot z^{a+1}}$. And e_1 and e_2 are known positive constants.

(iii) Hyperprior distributions of c_1, c_2, d_1^2, d_2^2 are given as followings;

$[c_1|g_1, h_1^2] \sim N(g_1, h_1^2)$,

$[c_2|g_2, h_2^2] \sim N(g_2, h_2^2)$,

$[d_1^2|p_1, q_1] \sim IG(p_1, q_1)$,

$[d_2^2|p_2, q_2] \sim IG(p_2, q_2)$,

where p_1 and p_2 are known positive constants.

To implement the Gibbs sampler, we need to calculate the full conditional distributions. From the hierarchical structure, the full conditional distributions are given as

(I) $[\mu_1|\mu_2, \sigma_1^2, \sigma_2^2, c_1, c_2, d_1^2, d_2^2, \underline{x}, \underline{y}] \propto exp(-\frac{\sum_{i=1}^m (x_i - \mu_1)^2}{2\sigma_1^2} - \frac{(\mu_1 - c_1)^2}{2d_1^2})$.

Where $\underline{x} = \{x_1, x_2, \cdots, x_m\}$ and $\underline{y} = \{y_1, y_2, \cdots, y_n\}$.

That is,

$$[\mu_1|\mu_2, \sigma_1^2, \sigma_2^2, c_1, c_2, d_1^2, d_2^2, \underline{x}, \underline{y}] \sim N(\frac{\sum_{i=1}^m x_i d_1^2 + c_1\sigma_1^2}{md_1^2 + \sigma_1^2}, \frac{\sigma_1^2 d_1^2}{md_1^2 + \sigma_1^2}). \quad (4)$$

(II) $[\mu_2|\mu_1, \sigma_1^2, \sigma_2^2, c_1, c_2, d_1^2, d_2^2, \underline{x}, \underline{y}] \propto exp(-\frac{\sum_{j=1}^n (y_j - \mu_2)^2}{2\sigma_2^2} - \frac{(\mu_2 - c_2)^2}{2d_2^2})$.

That is,

$$[\mu_2|\mu_1, \sigma_1^2, \sigma_2^2, c_1, c_2, d_1^2, d_2^2, \underline{x}, \underline{y}] \sim N(\frac{\sum_{j=1}^n y_j d_2^2 + c_2\sigma_2^2}{nd_2^2 + \sigma_2^2}, \frac{\sigma_2^2 d_2^2}{nd_2^2 + \sigma_2^2}). \quad (5)$$

(III) $[\sigma_1^2|\mu_1,\mu_2,\sigma_2^2,c_1,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \propto \frac{1}{\sigma_1^m\cdot\sigma_1^{2e_1+2}}\cdot exp(-\frac{\sum_{i=1}^m(x_i-\mu_1)^2}{2\sigma_1^2}-\frac{1}{\sigma_1^2 f_1})$.

That is,

$$[\sigma_1^2|\mu_1,\mu_2,\sigma_2^2,c_1,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \sim IG(\frac{m}{2}+e_1,\ (\frac{\sum_{i=1}^m(x_i-\mu_1)^2}{2}+\frac{1}{f_1})^{-1}). \quad (6)$$

(IV) $[\sigma_2^2|\mu_1,\mu_2,\sigma_1^2,c_1,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \propto \frac{1}{\sigma_2^n\cdot\sigma_1^{2e_2+2}}\cdot exp(-\frac{\sum_{j=1}^n(y_j-\mu_2)^2}{2\sigma_2^2}-\frac{1}{\sigma_2^2 f_2})$.

That is,

$$[\sigma_2^2|\mu_1,\mu_2,\sigma_1^2,c_1,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \sim IG(\frac{n}{2}+e_2,\ (\frac{\sum_{j=1}^n(y_j-\mu_2)^2}{2}+\frac{1}{f_2})^{-1}). \quad (7)$$

(V) $[c_1|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \propto exp(-\frac{(\mu_1-c_1)^2}{2d_1^2}-\frac{(c_1-g_1)^2}{2h_1^2})$.

That is,

$$[c_1|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_2,d_1^2,d_2^2,\underline{x},\underline{y}] \sim N(\frac{\mu_1 h_1^2+g_1 d_1^2}{h_1^2+d_1^2},\frac{d_1^2h_1^2}{h_1^2+d_1^2}). \quad (8)$$

(VI) $[c_2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,d_1^2,d_2^2,\underline{x},\underline{y}] \propto exp(-\frac{(\mu_2-c_2)^2}{2d_2^2}-\frac{(c_2-g_2)^2}{2h_2^2})$.

That is,

$$[c_2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,d_1^2,d_2^2,\underline{x},\underline{y}] \sim N(\frac{\mu_2 h_2^2+g_2 d_2^2}{h_2^2+d_2^2},\frac{d_2^2h_2^2}{h_2^2+d_2^2}). \quad (9)$$

(VII) $[d_1^2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,c_2,d_2^2,\underline{x},\underline{y}] \propto \frac{1}{d_1^{3+2p_1}}exp(-\frac{(\mu_1-c_1)^2}{2d_1^2}-\frac{1}{d_1^2q_1})$.

That is,

$$[d_1^2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,c_2,d_2^2,\underline{x},\underline{y}] \sim IG(\frac{1}{2}+p_1,(\frac{(\mu_1-c_1)^2}{2}+\frac{1}{q_1})^{-1}). \quad (10)$$

(VIII) $[d_2^2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,c_2,d_1^2,\underline{x},\underline{y}] \propto \frac{1}{d_2^{3+2p_2}}exp(-\frac{(\mu_2-c_2)^2}{2d_2^2}-\frac{1}{d_2^2q_2})$.

That is,

$$[d_2^2|\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,c_1,c_2,d_1^2,\underline{x},\underline{y}] \sim IG(\frac{1}{2}+p_2,(\frac{(\mu_2-c_2)^2}{2}+\frac{1}{q_2})^{-1}). \quad (11)$$

To implement the Gibbs sampler, we should be able to draw samples from the conditional densities given in (4)-(11). Simulation from the conditional densities can be done by standard methods.

In implementing Gibbs sampler in our problems, we follow the recommendation of Gelman and Rubin(1992). By Gelman Rubins' algorithm, independently simulate $l \geq 2$ sequences, each of length $2t$, with starting points drawn from an overdispersed distribution. To diminish the effect of the starting distribution, discard the first t iterations of each sequence, and focus attention on the last t.

Hence, we obtain Bayes estimators of $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ and θ as following;

$$E(\widehat{\mu_1|\underline{x}}) \approx \frac{1}{lt}\sum_{j=1}^{l}\sum_{k=t+1}^{2t}[\mu_1|\mu_{2j}^{(k)}, {\sigma_{1j}}^{2(k)}, {\sigma_{2j}}^{2(k)}, m_{1j}^{(k)}, m_{2j}^{(k)}, {d_{1j}}^{2(k)}, {d_{2j}}^{2(k)}, \underline{x}, \underline{y}]. \quad (12)$$

$$E(\widehat{\mu_2|\underline{y}}) \approx \frac{1}{lt}\sum_{j=1}^{l}\sum_{k=t+1}^{2t}[\mu_2|\mu_{1j}^{(k)}, {\sigma_{1j}}^{2(k)}, {\sigma_{2j}}^{2(k)}, m_{1j}^{(k)}, m_{2j}^{(k)}, {d_{1j}}^{2(k)}, {d_{2j}}^{2(k)}, \underline{x}, \underline{y}]. \quad (13)$$

$$E(\widehat{\sigma_1^2|\underline{x}}) \approx \frac{1}{lt}\sum_{j=1}^{l}\sum_{k=t+1}^{2t}[\sigma_1^2|\mu_{1j}^{(k)}, \mu_{2j}^{(k)}, {\sigma_{2j}}^{2(k)}, m_{1j}^{(k)}, m_{2j}^{(k)}, {d_{1j}}^{2(k)}, {d_{2j}}^{2(k)}, \underline{x}, \underline{y}]. \quad (14)$$

$$E(\widehat{\sigma_2^2|\underline{y}}) \approx \frac{1}{lt}\sum_{j=1}^{l}\sum_{k=t+1}^{2t}[\sigma_2^2|\mu_{1j}^{(k)}, \mu_{2j}^{(k)}, {\sigma_{1j}}^{2(k)}, m_{1j}^{(k)}, m_{2j}^{(k)}, {d_{1j}}^{2(k)}, {d_{2j}}^{2(k)}, \underline{x}, \underline{y}]. \quad (15)$$

Now, we can see that the hierarchical Bayesian analysis of R is equivalent to that of θ since $\Phi(\cdot)$ is one-to-one function. Hence we must to obtain Bayes estimator of θ to obtain Bayesian estimator of R.

To estimate the posterior distribution of θ, it is necessary to find the full conditional distribution of θ.

With Gibbs sequence from the full conditional distributions, we can obtain

$$\theta_j^{(k)} = \frac{\mu_{1j}^{(k)} - \mu_{2j}^{(k)}}{\sqrt{\sigma_{1j}^{2(k)} + \sigma_{2j}^{2(k)}}}, \; j = 1, 2, \cdots, l, \; k = t+1, 2, \cdots, 2t. \quad (16)$$

Then the $\theta_j^{(k)}$ can be regard as samples from unknown postorior distribution of θ because of continuity of θ.

From the Gibbs sampler procedure, if $\{\theta_1^{t+1}, \cdots, \theta_l^{2t}\}$ is a sample from posterior of θ, Bayes estimator of the θ is approximated by

$$\hat{\theta} \approx \frac{1}{lt}\sum_{j=1}^{l}\sum_{k=t+1}^{2t}\frac{\mu_{1j}^{(k)} - \mu_{2j}^{(k)}}{\sqrt{\sigma_{1j}^{2(k)} + \sigma_{2j}^{2(k)}}} \quad (17)$$

Table 1: Rocket-motor Experiment Data

Chamber burst strength (X)	Operating pressure (Y)
15.30 17.10 16.30	7.74010 7.77490 7.72270
16.05 16.75 16.60	7.77925 7.96195 7.44720
17.10 17.50 16.10	8.07070 7.89525 8.07360
16.10 16.00 16.75	7.49650 7.57190 7.79810
17.50 16.50 16.40	7.87640 8.19250 8.01705
16.00 16.20	7.94310 7.71835 7.87785
	7.29040 7.75750 7.31960
	7.63570 8.06055 7.91120

Also, the 95 percent credibility interval(equal tails) is

$$(\theta_{[0.025\times lt]}\ ,\ \theta_{[0.975\times lt]}), \tag{18}$$

where $[0.025 \times lt]$ and $[0.975 \times lt]$ are the $(0.025 \times lt)$th and $(0.975 \times lt)$th order statistics of θ.(See Dey and Lee(1992)).

We can check the monitor convergence of the iterative simulation by Gelman and Rubin's statistics(See Gelman and Rubin(1992)), which declines to 1 as $t \longrightarrow \infty$.

4. Illustration

In this Section, an illustrative example is represented by the rocket-motor experiment data reported by Guttman, Johnson, Bhattacharyya, and Reiser (1988). Suppose that one is interested in testing the reliability of the rocket motor at the highest operating temperature-namely, 59 degrees centigrade-at which the operating pressure (Y) distribution tends to be closest to the chamber burst strength (X) distribution. As in the work of Guttman et al., each distribution is assumed to be normal; a quantile-quantile plot of data has supported this assumption. Shown in Table 1 are some observed values of 17 motor cases and a sample size 24 from the operating pressure distribution.

The observed summary statistics are $\overline{x} = 16.485$, $\overline{y} = 7.789$, $s_1^2 = 0.3409$, $s_2^2 = 0.05414$, and $\hat{\theta} = 13.836$, where s_1^2 and s_2^2 are sample variances of random variables X and Y, respectively. Also an approximate 95 percent confidence interval for θ is $[9.52,\ 17.67]$.

In Gibbs sampling approach, we place diffuse second stage prior on $p_1 = 1, q_1 = 10^{-5}, p_2 = 1, q_2 = 10^{-5}$ and $h_1^2 = h_2^2 = 10000$.

To implement the Gibbs sampler, we consider $l = 3$ independent sequences each with a sample of size $2t = 3000$.

Figure 1 present estimated marginal posterior distribution of θ. From figure 1, we can see that the estimated marginal posterior distribution has unimodal and that Bayes estimate of θ is 13.4. Figures 2-3 present traces and autocorrelation function of θ out to lag 50. To check convergence of Gibbs sampler, figure 4 presents the plot of statistics proposed by Gelman and Rubin(1992). We can check that the statistics declines to 1. From figures 2-4, the Gibbs sampling algorithm seemed to achieve the convergence. The figures were obtained by use of Win BUGS software version 1.2.

As results of Gibbs sampling algorithm, Bayes estimate of the interest parameter θ is 13.4 and it's standard error is 2.136. Also, 95 percent Bayesian credible interval is (9.311, 17.52). The results of Bayes estimate are similar with nonBayes estimate given by Guttman et.al.(1988). In spite of small sample size, Bayesian viewpoint provide estimated probability distribution function, standard error, and interval estimate, respectively. But, nonBayesian viewpoint by Guttman et.al. provide standard error, interval estimate, and estimated probability distribution function only for large sample.

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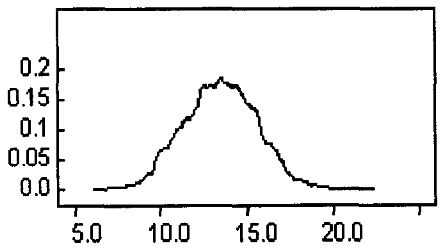

Figure 1: Estimated Marginal Posterior Density of θ

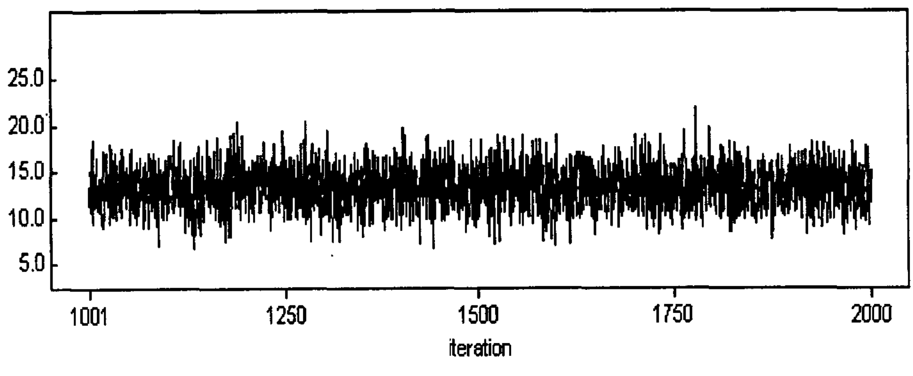

Figure 2: Traces of θ

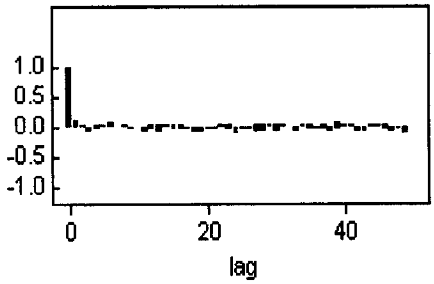

Figure 3: Autocorrelation Function of θ Out to Lag 50

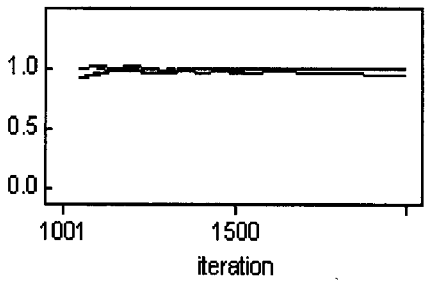

Figure 4: Gelman and Rubin's Statistics for θ