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AMLE for the Gamma Distribution under the Type-I censored sample

Suk-Bok Kang[1] · Hwa-Jung Lee[2]

Abstract

By assuming a Type-I censored sample, we propose the approximate maximum likelihood estimators(AMLE) of the scale and location parameters of the gamma distribution. We compare the proposed estimators with the maximum likelihood estimators(MLE) in the sense of the mean squared errors(MSE) through Monte Carlo method.

Key Words and Phrases: Approximate maximum likelihood estimator(AMLE), Gamma distribution, Type-I censored sample

1. Introduction

The probability density function of the random variable X having a gamma distriubtion with location parameter γ, the scale parameter β, and the shape parameter α is given by

$$f(x) = \frac{(x-\gamma)^{\alpha-1}\exp[-(x-\gamma)/\beta]}{\beta^{\alpha}\Gamma(\alpha)}, \qquad \alpha > 0, \;\; \beta > 0, \;\; x > \gamma. \tag{1.1}$$

Like the Weibull, lognormal and inverse Gaussian distribution, the gamma distribution is another positively skewed distribution that is frequently employed as a model for life spans, reaction times, and related phenomena. Especially the three-parameter gamma distribution is well known as type III in the Pearson system of frequency distribution.

[1]Professor, Department of Statistics and Institute of Natural Science, Yeungnam University, 214-1 Daedong, Kyongsan, Kyongbuk, 712-749, South Korea
[2]Graduate, Department of Statistics, Yeungnam University, 214-1 Daedong, Kyongsan, Kyongbuk, 712-749, South Korea

In most cases of censored sample, the explicit estimators may be not obtained by the maximum likelihood method. So we need another method that generates the explicit estimator. The approximate maximum likelihood estimating method was first developed by Balakrishnan (1989a, b) for the purpose of providing the explicit estimators of the scale parameter in the Rayleigh distribution and the mean and standard deviation in the normal distribution with censoring. Balakrishnan and Wong (1989) obtained approximate maximum likelihood estimators(AMLE) of the location and scale parameters in the half-logistic distribution with Type II-right censoring. Balakrishnan (1990) obtained AMLEs of the location and scale parameters in the generalized logistic distribution. Balakrishnan and Varadan (1990) obtained AMLEs of the location and scale parameters in the extreme value distribuion with censoring. Kang (1996) obtained the AMLE for the scale parameter of the double exponential distribution based on Type-II censored samples and he showed that the proposed estimator is generally more efficient than the best linear unbiased estimator and the optimum unbiased absolute estimater. Kang et al. (1997) proposed the minimum risk estimator(MRE) and the AMLE of the location and scale parameters of the two-parameter exponential distribution with Type-II censoring. Kang and Cho (1997) proposed the AMLE of the scale parameter of the one-parameter exponential distribution under general progressive Type-II censored samples. Woo et al. (1998) obtained the AMLE of the scale parameter of the p-dimensional Rayleigh distribution with singly right censored samples.

In this paper, we propose the estimators of the scale and location parameters in the gamma distribution with censored samples using the approximate maximum likelihood method when the shape parameter is known. We also compare the estimators in the sense of the MSE through the Monte Carlo simulation.

2. Estimation based on the Type-I right censored sample

The cumulative distribution function(cdf) of the gamma distribution with pdf (1.1) is given by

$$F(x) = [\Gamma(\alpha)]^{-1}\int_{\gamma}^{x}\Big(\frac{t-\gamma}{\beta}\Big)^{\alpha-1}\exp\Big(-\frac{t-\gamma}{\beta}\Big)dt.$$

The right censored samples(Cohen, 1991) consist of a total of n observations of which r are fully measured while $c = n - r$ are censored. Suppose it is decided to terminate the experiment at a predetermined time T in the Type-I censored sample. For each of the censored observations, it is known only that $T \le x$, whereas for each of the measured observations, $x \le T$ where T is a fixed (known) point of censoring.

Now we will obtain the estimators of the location parameter and the scale parameter. First, we use the $X_{1:n}$ as the estimator of the location parameter. Then we shall obtain an AMLE of the scale parameter.

The loglikelihood function based on Type-I censored sample is given by

$$\ln L = \ln K - n\alpha\ln\beta - \ln\Gamma(\alpha) + \sum_{i=1}^{n} \ln f(x_{i:n}) + c\ln[1 - F(T)],$$

where K is a constant.

Using the $\hat{\gamma} = X_{1:n}$, we will obtain the estimator of the scale parameter β by the approximate maximum likelihood estimation method.

The maximum likelihood equation for β is

$$\frac{\partial \ln \mathrm{L}}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_{i:n} - \gamma) - \frac{c}{1 - F(T)}\frac{\partial F(T)}{\partial \beta} = 0. \tag{2.1}$$

When we make the standardizing transformation $Z = \frac{X - E(X)}{\sqrt{\mathrm{Var}(X)}} = \frac{X - \gamma - \alpha\beta}{\beta\sqrt{\alpha}}$, the standard distribution with α as the shape parameter becomes

$$g(z) = \begin{cases} \frac{(\sqrt{\alpha})^{\alpha}}{\Gamma(\alpha)}(z + \sqrt{\alpha})^{\alpha - 1}\exp\{-\sqrt{\alpha}(z + \sqrt{\alpha})\}, & -\sqrt{\alpha} \le z < \infty, \\ 0, & otherwise. \end{cases}$$

Let $\frac{T - \gamma}{\beta\sqrt{\alpha}} - \sqrt{\alpha} = y$, then

$$F(T) = G(y) = \int_{-\sqrt{\alpha}}^{y} g(t)dt,$$

where G is the cdf of z.

The partial derivative of F with respect to β may be expressed as

$$\frac{\partial F(T)}{\partial \beta} = \frac{\partial G(y)}{\partial y}\frac{\partial y}{\partial \beta} = -\frac{1}{\beta}(y + \sqrt{\alpha})g(y). \tag{2.2}$$

When the result (2.2) is substituted into (2.1), the estimating equations becomes

$$\frac{\partial \ln \mathrm{L}}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_{i:n} - \gamma) + \frac{c(y + \sqrt{\alpha})}{\beta}\frac{g(y)}{1 - G(y)} = 0. \tag{2.3}$$

We can expand the function $g(y)/(1 - G(y))$ to Taylor series around the points p and then approximate it by

$$\frac{g(y)}{1-G(y)} \simeq \eta_1 + \eta_2 y, \tag{2.4}$$

where

$$\eta_1 = \frac{g(p)}{1-G(p)} - \frac{g'(p)[1-G(p)] + g(p)^2}{[1-G(p)]^2} p,$$

and

$$\eta_2 = \frac{g'(p)[1-G(p)] + g(p)^2}{[1-G(p)]^2}.$$

Now making use of the approximate expressions in (2.4), we may approximate the likelihood equation (2.3) as follows;

$$\frac{\partial \text{lnL}}{\partial \beta} \simeq \frac{\partial \text{lnL}^*}{\partial \beta} = -\frac{n\alpha}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n}(x_{i:n} - \gamma) + \frac{c(y+\sqrt{\alpha})}{\beta}(\eta_1 + \eta_2 y) = 0.$$

Upon solving equation for β, we derive the AMLE of β as follows;

$$\hat{\beta}_{AMLE} = \frac{-B + \sqrt{B^2 + 4nc\eta_2(T-\hat{\gamma})^2}}{2n\alpha}$$

where

$$B = -\sum_{i=1}^{n}(x_{i:n} - \hat{\gamma}) - \frac{c\,(T-\hat{\gamma})\,\eta_1}{\sqrt{\alpha}} + c\,(T-\hat{\gamma})\,\eta_2.$$

To compare the estimators of β, we simulate the mean squared errors of the AMLE and MLE through the Monte Carlo method. The simulation procedure is repeated 1,000 times for sample size $n = 20(10)50$. Censored time T_i have three cases. This simulated values are given in Table 1. From Table 1, the MSEs of the AMLE is smaller than the MSEs of the MLE in the $\beta = 0.5$ and $\alpha = 0.6$. But the MLE is more efficient than the AMLE in the other cases.

3. The Simulated Results

Random numbers of the gamma distribution were generated by IMSL subroutine RNGAM. The MLEs of γ and β, $\hat{\gamma}$ and $\hat{\beta}_{MLE}$, were obtained by Binary-search method. We simulate the MSEs of the estimators through the Monte method. The simulation procedure is repeated 1,000 times in the Type-I right censored sample for $\beta = 0.5$, 1.0, 2.0 and sample sizes $n = 20(10)50$. Censored time T_1 almost does not have censoring samples. T_2 is cenored about $10 \sim 20$ percentile and T_3 is about $40 \sim 50$ percentile from complete samples. These values are given in Table 1.

The MSE of the MLE is smaller than the MSE of the AMLE in the bell-shape($\beta = 1.5$), but the MSE of the AMLE is smaller than the MSE of the MLE in the $\beta = 0.5$ and $\alpha = 0.6$. AMLE has the advantage of explicit form and it is easy to estimate the scale and location parameter by using the AMLE.

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Table 1. The MSEs of the estimators for the location and scale parameters based on the Type-I censored sample.

$\gamma = 5$ and $\alpha = 0.6$

β	T	n	$\hat{\gamma}$	$\hat{\beta}_{AMLE}$	$\hat{\beta}_{MLE}$
0.5	6.8	20	0.000068	0.021338	0.020555
		30	0.000017	0.013739	0.013726
		40	0.000008	0.010357	0.010585
		50	0.000004	0.008051	0.008242
	5.6	20	0.000068	0.022232	0.025364
		30	0.000017	0.014763	0.017520
		40	0.000008	0.010965	0.013703
		50	0.000004	0.008404	0.010756
	5.2	20	0.000068	0.032099	0.039078
		30	0.000017	0.022882	0.029957
		40	0.000008	0.019467	0.023341
		50	0.000004	0.016890	0.018429
1.0	9	20	0.000272	0.084942	0.059875
		30	0.000067	0.054717	0.045701
		40	0.000031	0.041118	0.037975
		50	0.000016	0.032048	0.030755
	6	20	0.000272	0.090353	0.070512
		30	0.000067	0.060619	0.054178
		40	0.000031	0.045636	0.046727
		50	0.000016	0.034510	0.038093
	5.5	20	0.000272	0.112749	0.078430
		30	0.000067	0.078798	0.064750
		40	0.000031	0.064613	0.057505
		50	0.000016	0.053855	0.047534
2.0	12.0	20	0.001086	0.341884	0.102645
		30	0.000267	0.219195	0.091650
		40	0.000125	0.165279	0.083796
		50	0.000063	0.128517	0.076898
	7.5	20	0.001086	0.364979	0.107999
		30	0.000267	0.238297	0.097674
		40	0.000125	0.176631	0.091155
		50	0.000063	0.135019	0.082430
	6.0	20	0.001086	0.450994	0.116568
		30	0.000267	0.315192	0.106778
		40	0.000125	0.258452	0.100626
		50	0.000063	0.215420	0.092850

Table 1. (Continued)

$\gamma = 5$ and $\alpha = 1.5$

β	T	n	$\hat{\gamma}$	$\hat{\beta}_{AMLE}$	$\hat{\beta}_{MLE}$
0.5	8.0	20	0.009991	0.007813	0.007698
		30	0.006066	0.005335	0.005271
		40	0.004163	0.004058	0.004022
		50	0.002904	0.003216	0.003187
	6.5	20	0.009991	0.011860	0.009350
		30	0.006066	0.007448	0.006330
		40	0.004163	0.005486	0.004758
		50	0.002904	0.004316	0.003750
	5.8	20	0.009991	0.026167	0.013709
		30	0.006066	0.014745	0.009315
		40	0.004163	0.010747	0.006866
		50	0.002904	0.008546	0.005372
1.0	11.0	20	0.039964	0.031250	0.028118
		30	0.024262	0.021339	0.020148
		40	0.016650	0.016233	0.015875
		50	0.011617	0.012865	0.012670
	8.0	20	0.039964	0.047442	0.032564
		30	0.024262	0.029793	0.023607
		40	0.016650	0.021944	0.018302
		50	0.011617	0.017262	0.014681
	6.5	20	0.039964	0.115866	0.047670
		30	0.024262	0.063266	0.036100
		40	0.016650	0.046977	0.027999
		50	0.011617	0.038094	0.022523
2.0	17.0	20	0.159858	0.125001	0.072629
		30	0.097049	0.085357	0.059706
		40	0.066600	0.064933	0.051883
		50	0.046470	0.051461	0.044882
	11.0	20	0.159858	0.189767	0.078160
		30	0.097049	0.119171	0.067200
		40	0.066600	0.087777	0.058349
		50	0.046470	0.069048	0.049424
	9.0	20	0.159858	0.283361	0.087916
		30	0.097049	0.172096	0.075969
		40	0.066600	0.124771	0.065628
		50	0.046470	0.096893	0.057534