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Lindley Type Estimators with the Known Norm [1]

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Abstract

Consider the problem of estimating a $p \times 1$ mean vector $\underline{\theta}(p \geq 4)$ under the quadratic loss, based on a sample $\underline{x}_1, \cdots \underline{x}_n$. We find an optimal decision rule within the class of Lindley type decision rules which shrink the usual one toward the mean of observations when the underlying distribution is that of a variance mixture of normals and when the norm $\| \underline{\theta} - \bar{\theta}\underline{1} \|$ is known, where $\bar{\theta} = (1/p)\sum_{i=1}^{p} \theta_i$ and $\underline{1}$ is the column vector of ones.

Key Words and Phrases: Lindley type decision rule, Mean vector, Quadratic loss, Optimal decision rule, Underlying distribution

1. Introduction

The problem considered is that of estimating with quadratic loss function the mean vector of a compound multinormal distribution when the norm $\| \underline{\theta} - \bar{\theta}\underline{1} \|$ is known. The class of estimation rules considered will consist of Lindley type estimators only. Such a class was introduced by James-Stein (1961) and Lindley (1962) in order to prove that some of its members dominate the sample mean in the multinormal case. Strawderman (1974) also derived a similar result for the more general case considerd in this paper of a compound multinormal distribution.

The paroblem of estimation of a mean under constraint has an old origin and recently focussed again in the context of curved model in the works of Efron (1975), Hinckley (1977), Amari (1982), Kariya (1989), Perron and Giri (1990), Marchand and Giri (1993) among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger (1975).

In Section 2, we present the general setting of our problem and develop necessary notations. In Section 3, we derive the best Lindley type estimator of a mean when the norm $\| \underline{\theta} - \bar{\theta}\underline{1} \|^2$ is known. Examples of these best estimators are given in Section 4.

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2. Notation and Preliminaries

Let $\underline{X} = (X_1, \cdots, X_p)$, $p \geq 4$ be an observation from a compound multinormal distribution with unknown location parameter $\underline{\theta}(p\times 1)$ and mixture parameter $H(\cdot)$, where $H(\cdot)$ represents a known c.d.f. defined on the interval $(0, \infty)$. In other words, we assume that the random variable $\underline{X}$ generating our observation $\underline{x}$ admits the representation,

$$\mathcal{L}(\underline{X}|Z = z) = N_p(\underline{\theta},\ zI_p),\ \forall z > 0\ , \tag{2.1}$$

Z being the positive random variable with c.d.f. $H(\cdot)$.

Our problem concerns the estimation of the location parameter $\underline{\theta}$ with loss function

$$L(\underline{\theta},\ \delta(\underline{x})) = (\delta(\underline{x}) - \underline{\theta})'(\delta(\underline{x}) - \underline{\theta}),$$

with $\underline{\theta} \in \Theta_\lambda = \{\underline{\theta} \in \mathbb{R}^p|\ \| \underline{\theta} - \bar{\theta}\underline{1} \| = \lambda,\ 0 \leq \lambda < \infty\}$, where $\bar{\theta} = \frac{1}{p}\sum_1^p \theta_i$, $\underline{1}' = (1, \cdots, 1)$ and the decision rule δ, $\delta(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$, is of the form

$$\delta(\underline{x}) = \bar{x}\underline{1} + \left(1 - \frac{c}{(\underline{x} - \bar{x}\underline{1})'(\underline{x} - \bar{x}\underline{1})}\right)(\underline{x} - \bar{x}\underline{1})\ ,\quad c \in \mathbb{R}\ .$$

Restated in terms of the family of probability density functions of $\underline{X}$, the distributional assumption given by expression (2.1) and the restriction on the location parameter $\underline{\theta}$ indicate that the p.d.f. of $\underline{X}$ is

$$p_{\underline{\theta}}(\underline{x}) = \int_{(0,\infty)} (2\pi z)^{-p/2} exp\left(\frac{-\| \underline{x} - \underline{\theta} \|^2}{2z}\right) dH(z)\ , \tag{2.2}$$

$\underline{x} \in \mathbb{R}^p$ and $\underline{\theta} \in \Theta_\lambda$. It will be also assumed that $E(Z) < \infty$ which will guarantee the existence of the covariance matrix $\sum = Cov(\underline{X}) = E(Z)I_p$ and the mean vector $E(\underline{X}) = \underline{\theta}$. The performance of the estimator δ will be measured by its risk function

$$R(\underline{\theta}, \delta) = E_\theta[L(\underline{\theta}, \delta(\underline{X}))] = E_\theta[(\delta(\underline{X}) - \underline{\theta})'(\delta(\underline{X}) - \underline{\theta})],\ \underline{\theta} \in \Theta_\lambda\ .$$

3. Optimal Lindley Estimation when the norm $\| \underline{\theta} - \bar{\theta}\underline{1} \|$ is known

In this section, the best estimator is derived within

$$\mathcal{D}_{Lind} = \{\delta : \mathbb{R}^p \rightarrow \mathbb{R}^p|\delta(\underline{x}) = \delta^c(\underline{x}) = \bar{x}\underline{1} + \left(1 - \frac{c}{(\underline{x} - \bar{x}\underline{1})'(\underline{x} - \bar{x}\underline{1})}\right)(\underline{x} - \bar{x}\underline{1}),\ c \in \mathbb{R}\},$$

where the parameter space is of the form

$$\Theta_\lambda = \{\underline{\theta} \in \mathbb{R}^p | \parallel \underline{\theta} - \bar{\theta}\underline{1} \parallel = \lambda\}, \quad \lambda \geq 0 \,.$$

The following lemmas will prove useful in the evaluation of the risk function of the decision rule δ^c, $c \in \mathbb{R}$.

Lemma 3.1. Let $\underline{X}$ be a random multinormal vector $N_p(\underline{\theta}, I_p)$, $p \geq 4$ and $\underline{\theta} \in \mathbb{R}^p$. Then

$$\text{(i) } \mathrm{E}_{\underline{\theta}}\left(\frac{1}{(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}\right) = \mathrm{E}^{\mathrm{L}}\left(\frac{1}{\mathrm{p} + 2\mathrm{L} - 3}\right)$$

and

$$\text{(ii) } \mathrm{E}_{\underline{\theta}}\left(\frac{(\underline{\mathrm{X}} - \underline{\theta})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}{(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}\right) = \mathrm{E}^{\mathrm{L}}\left(\frac{\mathrm{p} - 3}{\mathrm{p} + 2\mathrm{L} - 3}\right),$$

where L is a Poisson random variable with mean $(\underline{\theta} - \bar{\theta}\underline{1})'(\underline{\theta} - \bar{\theta}\underline{1})/2$

Proof. See James and Stein(1961) and use Stein's Identity

Lemma 3.2. Let $\underline{X}$ be a compound multinormal vector with location parameter $\underline{\theta}; p \geq 4$ and $\underline{\theta} \in \mathbb{R}^p$; and known mixture parameter $H(\cdot)$ with p.d.f. of the form given in (2.2). Then, with $\lambda = \parallel \underline{\theta} - \bar{\theta}\underline{1} \parallel$

$$\text{(i) } \mathrm{E}_{\underline{\theta}}\left(\frac{1}{(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}\right) = \int_{(0,\infty)} \mathrm{f_p}(\lambda,\ \mathrm{z})\frac{\mathrm{dH(z)}}{\mathrm{z}} \,,$$

and

$$\text{(ii) } \mathrm{E}_{\underline{\theta}}\left(\frac{(\underline{\mathrm{X}} - \underline{\theta})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}{(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})'(\underline{\mathrm{X}} - \bar{\mathrm{X}}\underline{1})}\right) = (\mathrm{p} - 3)\int_{(0,\infty)} \mathrm{f_p}(\lambda,\ \mathrm{z})\mathrm{dH(z)} \,,$$

where the function $f_p(\cdot, \cdot) : [0, \infty) \to (0, \infty)$, is defined by the relation

$$f_p(\lambda,\ z) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p + 2j - 3)} \,.$$

Proof. (i) Using both the representation given in (2.1) and part(i) of Lemma 3.1, we obtain

$$\begin{aligned}
E_{\underline{\theta}}\left(\frac{1}{(\underline{X} - \bar{X}\underline{1})'(\underline{X} - \bar{X}\underline{1})}\right) &= E^Z\left\{Z^{-1}E_{\underline{\theta}}^{X|Z}\left[\frac{Z}{(\underline{X} - \bar{X}\underline{1})'(\underline{X} - \bar{X}\underline{1})}\right]\right\} \\
&= \int_{(0.\infty)} z^{-1}\sum_{j=1}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}\left(\frac{\lambda^2}{2z}\right)^j}{j!(p + 2j - 3)}dH(z) \\
&= \int_{(0,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z} \,.
\end{aligned}$$

(ii) Again, combining the representation given in (2.1) and part(ii) of Lemma 3.1, we obtain

$$
\begin{aligned}
E_{\underline{\theta}}\Big(\frac{(\underline{X}-\underline{\theta})'(\underline{X}-\bar{X}\underline{1})}{(\underline{X}-\bar{X}\underline{1})'(\underline{X}-\bar{X}\underline{1})}\Big) &= E^{Z}\Big\{E_{\underline{\theta}}^{X|Z}\Big[\frac{(\frac{\underline{X}-\underline{\theta}}{\sqrt{Z}})'(\frac{\underline{X}-\bar{X}\underline{1}}{\sqrt{Z}})}{(\frac{\underline{X}-\bar{X}\underline{1}}{\sqrt{Z}})'(\frac{\underline{X}-\bar{X}\underline{1}}{\sqrt{Z}})}\Big]\Big\} \\
&= \int_{(0,\infty)}\sum_{j=0}^{\infty}\frac{e^{-\frac{\lambda^2}{2z}}(\frac{\lambda^2}{2z})^j}{j!}\frac{p-3}{p+2j-3}dH(z) \\
&= (p-3)\int_{(0,\infty)} f_p(\lambda,\ z)dH(z)\ .
\end{aligned}
$$

The main result of this section now follows.

Theorem 3.3. Let $\underline{X}$ be a single observation from a p-dimensional location parameter with p.d.f. of the form given by (2.2). Under the assumptions $\underline{\theta} \in \Theta_\lambda$, $p \geq 4$ and $E[Z] < \infty$, the unique best estimator within the class $\mathcal{D}_{Lind}$ is given by $\delta^{c^*(\lambda)}$ where

$$
c^*(\lambda) = (p-3)\frac{\int_{(0,\infty)} f_p(\lambda,z)dH(z)}{\int_{(0,\infty)} f_p(\lambda,z)\frac{dH(z)}{z}}\ . \tag{3.1}
$$

Proof. Under the assumptions above, we can easily derive the result $E_{\underline{\theta}}(\underline{X}'\underline{X}) = \underline{\theta}'\underline{\theta} + pE(Z)$. Combining this with Lemma 3.2, we have

$$
\begin{aligned}
& R(\underline{\theta},\ \delta^c) \\
= \;& E_{\underline{\theta}}[(\delta^c(\underline{X})-\underline{\theta})'(\delta^c(\underline{X})-\underline{\theta})] \\
= \;& pE(Z) + \Big[c^2E_\theta\Big\{\frac{1}{(\underline{X}-\bar{X}\underline{1})'(\underline{X}-\bar{X}\underline{1})}\Big\} - 2cE_\theta\Big\{\frac{(\underline{X}-\underline{\theta})'(\underline{X}-\bar{X}\underline{1})}{(\underline{X}-\bar{X}\underline{1})'(\underline{X}-\bar{X}\underline{1})}\Big\}\Big] \\
= \;& pE(Z) + \Big[c^2\int_{(o,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z} - 2c(p-3)\int_{(o,\infty)} f_p(\lambda,\ z)dH(z)\Big] \\
= \;& pE(Z) + \Big\{\int_{(o,\infty)}\Big[\frac{c^2}{z} - 2c(p-3)\Big]f_p(\lambda,\ z)dH(z)\Big\}\ .
\end{aligned} \tag{3.2}
$$

From this last equality, we obtain easily that

$$
inf_{c\in R}R(\underline{\theta},\ \delta^c) = R(\underline{\theta},\ \delta^{c^*(\lambda)})
$$

with $c^*(\lambda)$ given by expression (3.1).

Using expression (3.2), the minimum risk attained by the best Lindley type estimator is equal to

$$R(\underline{\theta},\ \delta^{c^*(\lambda)}) = pE(Z) - (p-3)^2 \frac{[\int_{(0,\infty)} f_p(\lambda,\ z)dH(z)]^2}{\int_{(0,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}}\ , \quad \underline{\theta} \in \Theta_\lambda\ .$$

When $\| \underline{\theta} - \bar{\theta}\underline{1} \| = \lambda$, the use of other estimators of the Lindley class other than $\delta^{c^*(\lambda)}$ will incur risk which is a strictly increasing function of distance $|c - c^*(\lambda)|$. To see this, we can define $h(\lambda)$ such that $c = h(\lambda)c^*(\lambda)$ and, using expression (3.2), express $R(\underline{\theta},\ \delta^c)$ as

$$pE(Z) + (p-3)^2[h^2(\lambda) - 2h(\lambda)]\frac{[\int_{(o,\infty)} f_p(\lambda,\ z)dH(z)]^2}{\int_{(o,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}}\ . \tag{3.3}$$

From this we can write

$$R(\underline{\theta},\ \delta^c) - R(\underline{\theta},\ \delta^{c^*(\lambda)}) = |c - c^*(\lambda)|^2 \int_{(o,\infty)} f_p(\lambda,\ z)\frac{dH(z)}{z}\ . \tag{3.4}$$

The natural estimator $\delta^o(\underline{x}) = \underline{x}$ is a member of the Lindley class and has a constant risk function equal to $pE(Z)$. We can also characterize the estimators of the Lindley type that dominate the natural estimator δ^o.

Corollary 3.4. Under the conditions of Theorem 3.3, the decision rule δ^c will dominate the natural estimator δ^o if and only if $0 < c < 2c^*(\lambda)$.
Proof. Using expression (3.3), one easily sees that, for $\theta \in \Theta_\lambda$,

$$\begin{aligned} R(\underline{\theta},\ \delta^c) &< R(\underline{\theta},\ \delta^o) = pE(Z) \\ &\Leftrightarrow h^2(\lambda) - 2h(\lambda) < 0 \\ &\Leftrightarrow 0 < h(\lambda) < 2 \\ &\Leftrightarrow 0 < c < 2c^*(\lambda)\ . \end{aligned}$$

4. Examples

The class of compound multinormal distributions is quite large and, in this section, we present some examples of the evaluation of the best Lindley type estimator for different choices of the underlying distribution of $\underline{X}$ or, equivalently, of the mixture parameter $H(\cdot)$.

Example 4.1. For $\underline{X} \sim N_p(\theta,\ \sigma^2 I_p)$, $p \geq 4$, (i.e. $H(Z) = 1_{(\sigma^2,\infty)}(Z)$ with $1_A(\cdot)$ being the indicator function of the set A); we deduce from Theorem 3.3 that

$$c^*(\lambda) = (p-3)\frac{f_p(\lambda,\ \delta^2)}{f_p(\lambda,\ \sigma^2)/\sigma^2} = (p-3)\sigma^2 ,$$

and that the best estimator within the Lindley class $\mathcal{D}_{Lind}$ is equal to

$$\delta^{(p-3)\sigma^2}(\underline{x}) = \bar{x}\underline{1} + \left(1 - \frac{(p-3)\sigma^2}{(\underline{x}-\bar{x}\underline{1})'(\underline{x}-\bar{x}\underline{1})}\right)(\underline{x}-\bar{x}\underline{1}) ,$$

irregardless of the value of the norm $\lambda = \| \underline{\theta} - \bar{\theta}\underline{1} \|$.

For non-normal cases, the following explicit formula for the quantity $f_p^*(\gamma) = E^L[(p+2L-3)^{-1}]$, $L \sim \text{Poisson}(\gamma)$, given by Egerton and Laycock (1982) prove useful for the evaluation of the function $c^*(\lambda)$, $\lambda \geq 0$.

Lemma 4.1. Let L be a Poisson random variable with mean γ, $\gamma > 0$, and $f_p^*(\gamma) = E^L[(p+2L-3)^{-1}]$; $p \geq 4$; then

$$\text{(i) } \mathrm{f}_\mathrm{p}^*(\gamma) = \mathrm{e}^{-\gamma}\int_{[0,1]} \mathrm{t}^{\mathrm{p}-4}\mathrm{e}^{\gamma \mathrm{t}^2}\mathrm{dt} ,$$

and

$$\text{(ii) } \mathrm{f}_{\mathrm{p}+2}^*(\gamma) = (2\gamma)^{-1}[1-(\mathrm{p}-3)\mathrm{f}_\mathrm{p}^*(\gamma)] . \tag{4.1}$$

For even values of the dimension p, the recurrence formula given by expression (4.1) permits the expression of the function $f_p^*(\cdot)$ as a function of $f_4^*(\cdot)$. From part(i) of the preceding lemma,

$$\begin{aligned} f_4^*(\gamma) &= e^{-\gamma}\int_{[0,1]} e^{\gamma t^2}dt \\ &= \gamma^{-\frac{1}{2}}D(\gamma^{-\frac{1}{2}}) , \end{aligned}$$

where $D(x) = e^{-x^2}\int_{(0,x)} e^{t^2}dt$, $x > 0$, is known as Dawson's integral which is tabulated in Abramowitz and Stegun (1965). For odd values of the dimension p, the recurrence formula given by expression (4.1) permits the expression of the function $f_p^*(\cdot)$ as a function of $f_5^*(\cdot)$. From part(i) of Lemma 4.1,

$$\begin{aligned} f_5^*(\gamma) &= e^{-\gamma}\int_{[0,1]} te^{rt^2}dt \\ &= (2\gamma)^{-1}(1-e^{-\gamma}) . \end{aligned} \tag{4.2}$$

We now proceed with the evaluation of the best Lindley estimator in the contaminated multinormal case.

Example 4.2. Setting $H(z) = \sum_{j=1}^{n} \epsilon_j 1_{[\sigma_j^2,\infty)}(z)$ in expression (2.2), where $0 < \epsilon_j < 1$, $\sigma_j^2 > 0$ for $j \in \{1, \cdots, n\}$ and $\sum_{j=1}^{n} \epsilon_j = 1$, we obtain the family of contaminated multinormal distributions with mean parameter $\underline{\theta}$ and known dispersion parameters $(\sigma_1^2,\ \epsilon_1), \cdots, (\sigma_n^2,\ \epsilon_n)$. The function $c^*(\lambda)$, $\lambda \geq 0$, defined by (3.1) becomes

$$c^*(\lambda) = (p-3)\frac{\sum_{j=1}^{n} \epsilon_j f_p(\lambda,\ \sigma_j^2)}{\sum_{j=1}^{n} \frac{\epsilon_j}{\sigma_j^2} f_p(\lambda,\ \sigma_j^2)},$$

and the decision rule $\delta^{c^*(\lambda)}$ represents, by Theorem 3.3, the best Lindley type estimator when $\underline{\theta} \in \Theta_\lambda$. The quantities $f_p(\lambda,\ \sigma_j^2)$ can be evaluated by using the results of Lemma 4.1. In particular, for $p = 7$, using expressions (4.1) and (4.2), we obtain

$$\begin{aligned} f_7(\lambda,\ z) &= f_7^*\left(\frac{\lambda^2}{2z}\right) \\ &= \lambda^{-4} z(\lambda^2 - 2z + 2ze^{-\lambda^2/2z}),\quad \lambda > 0,\quad z > 0, \end{aligned}$$

and,

$$c^*(\lambda) = 4\,\frac{\sum_{j=1}^{n} \epsilon_j \sigma_j^2(\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}{\sum_{j=1}^{n} \epsilon_j(\lambda^2 - 2\sigma_j^2 + 2\sigma_j^2 e^{-\lambda^2/2\sigma_j^2})}.$$

Example 4.3. Setting $\mathcal{L}(z^{-1}) = \text{Gamma}(a, b)$, $a > 1$ and $b > 0$, in the representation given by expression (2.1), we obtain the family of multivariate student distributions with mean parameter θ (the condition $a > 1$ guaranteeing the existence of a covariance matrix) and known dispersion parameter (a, b). Here, we extend the usual class of multivariate student location families with n degrees of freedom, where $n = 2a = 2b$ and $n \in \{1, 2, \cdots\}$, to include other values of the dispersion parameter (a, b). For the particular case where $p = 5$, we obtain by expressions (3.1) and (4.2),

$$f_5(\lambda,\ z) = f_5^*\left(\frac{\lambda^2}{2z}\right) = \lambda^{-2} z(1 - e^{-\lambda^2/2z}),\quad \lambda > 0,\quad z > 0,$$

and

$$\begin{aligned} c^*(\lambda) &= 2\,\frac{\int_{(0,\infty)} z(1-e^{-\lambda^2/2z})dH(z)}{\int_{(0,\infty)}(1-e^{-\lambda^2/2z})dH(z)} \\ &= 2\,\frac{\int_{(0,\infty)}(v^{-1}-v^{-1}e^{-\lambda^2 v/2})v^{a-1}e^{-bv}dv}{\int_{(0,\infty)}(1-e^{-\lambda^2 v/2})v^{a-1}e^{-bv}dv} \\ &= \frac{2b}{a-1}\,\frac{\left[1-\left(\frac{2b}{2b+\lambda^2}\right)^{a-1}\right]}{\left[1-\left(\frac{2b}{2b+\lambda^2}\right)^{a}\right]}. \end{aligned}$$

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