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Notes on Upper and Lower Bounds of Odds Ratio

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Abstract

We shall give upper and lower bounds of the odds ratio of an event by a slight condition of the conditional probability of events.

Key Words and Phrases: odds ratio, conditional probalility.

1. Introduction

Throughout, all sets are used as events in the same sample space Ω. Define $\frac{P(A)}{P(A^c)}$ be the odds ratio of an event A, $0 < P(A) < 1$(see Ross(1998)). And in the Bernoulli trial, the odds ratio of a successive event can be applied to test whether the successive proportion equals to 1 or not. Sasienni(1994) studied confidence intervals and variance estimators of the odds ratio, and small-sample properties of a family of odds ratio estimators were studied by Sasienni(1994) and Ejigou(1990).

Here we shall give upper and lower bounds of the odds ratio of an event by a strongly more informationed condition of the conditional probability.

2. Main results

Let $P(A|M) = \frac{P(A\cap M)}{P(M)}$ be the conditional probability of A given M, $P(M) > 0$.

From definition of the conditional probability and the multiplication rule (Rohatgi (1976)), we can get it:

Fact 1.(Ross(1998)) (a) $P(A|M) = \frac{P(A)}{P(M)} \cdot P(M|A)$, if $P(A) > 0$ and $P(M) > 0$.
(b) $\frac{P(A)}{P(A^c)} = \frac{P(A|M)}{P(A^c|M)} \cdot \frac{P(M|A^c)}{P(M|A)}$, if all denominators are positive.

Proof. (b) comes from (a).

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From the result of Fact 1(a),

Fact 2. Let $0 < P(A) < 1$ and $0 < P(M) < 1$. Then

$$\begin{aligned} P(A|M) = P(M|A) \quad & iff \quad P(A) = P(M) \\ & iff \quad P(A^c) = P(M^c) \\ & iff \quad P(A^c|M^c) = P(M^c|A^c). \end{aligned}$$

It has been well-known in Ross(1998) that

$$\begin{aligned} \frac{P(M_2|M_1)}{P(M_2)} \geq 1 \quad & iff, \quad P(M_2|M_1) \geq P(M_2) \\ & iff, \quad M_1 \text{ carries positive information about } M_2. \\ \text{And the equality holds} \quad & iff, \quad M_1 \text{ and } M_2 \text{ are mutually independent.} \end{aligned}$$

It is very interesting that we shall compare the results of Fact 1(b) and Theorem 1.

Theorem 1. Let $0 < P(A) < 1$. Then
The odds ratio of an event A has a lower bound and an upper bound:

$$P(A|M) \cdot \frac{P(M|A^c)}{P(M|A)} < \frac{P(A)}{P(A^c)} < \frac{1}{P(A^c|M)} \cdot \frac{P(M|A^c)}{P(M|A)},$$

if all denominators are positive

Proof. From Bayes' formula, we get it :

$$P(A|M) = \frac{P(A)P(M|A)}{P(A)P(M|A) + P(A^c)P(M|A^c)} \tag{1.1}$$

Since $(1+\frac{1}{x})^{-1} < x$, for all positive x, then we can obtain the lower bound by letting $x = \frac{P(A)P(M|A)}{P(A^c)P(M|A^c)}$.

While, from Bayes formula (1.1), $P(A|M) = 1 - \frac{P(A^c)P(M|A^c)}{P(A)P(M|A)+P(A^c)P(M|A^c)}$. By the same method, we can obtain the upper bound by letting

$$x = \frac{P(A^c)P(M|A^c)}{P(A)P(M|A)}. \text{ So we have done(a).}$$

Let $P(A) > 0$. Then we define $P_A(\cdot) = P(\cdot|A)$ by the conditional probability function on the sample space(see Ross(1998))

Definition 1. If $P_A(M_2|M_1) = P_A(M_2)$, for $P(A) > 0$ and $P(M_1) > 0$, then M_2 is conditionally independent of M_1 with respect to A.

Example 1. Consider throwing a fair die.

Let $E = \{1,2,3,4\}$. $F = \{2,4,6\}$, $A = \{2,3,5\}$, $M = \{1,2,3,5\}$, and $G = \{2,3,5\}$.

Then E and F are mutually independent, but E and F are not conditonally independent with respect to A, and M and G are conditional independent with respect to M(or G), M and G are not mutually independent.

From definition of the condtional independent, we can obtain:

Fact 3. If $P_A(M_2|M_1) = P_A(M_2)$, then $P_A(M_1M_2) = P_A(M_1)P_A(M_2)$ and so M_1 and M_2 are mutually conditional independent with respect to A.

If an event A is the sure event, then M_1 and M_2 are well-known as mutually independent events, and it is clear thatM_1 and M_2 are conditionally independent with repect to $M_i, I = 1,2$, since $P_{M_i}(M_1M_2) = P_{M_i}(M_1)P_{M_i}(M_2)$ for every $i = 1,2$.

As we can extend the conditional independence of finite events as like the conditional independence of two events, we can obtain the followings:

Fact 4(Ross(1998)). (a) Let $E_1, E_2, ..., E_n$ be the conditional independent with respect to a positive event A. Then $P_A(\cup_{i=1}^n E_i) = 1 - \prod_{i=1}^n (1 - P_A(E_i))$.

(b) Let $\{E_n : n \geq 1\}$ and $\{F_n : n \geq 1\}$ be increasing sequences of events which $\lim E_n = E$ and $\lim F_n = F$, and the corresponding E_n and F_n be the conditional independent with respect to a positive event A. Then E and F are the conditional independent with respect to A.

Definition 2. Events M_1 and M_2 are more strongly informationed conditionally givenA^c than given A, if $\frac{P_{A^c}(M_2|M_1)}{P_{A^c}(M_2)} \geq \frac{P_A(M_2|M_1)}{P_A(M_2)}$, if all denominators are positive.

From its definition and Fact 3, we can know it easily.

Theorem 2. Events M_1 and M_2 are more strongly informationed conditionally given

$$A^c \text{ than given } A \text{ iff}, \frac{P_{A^c}(M_1M_2)}{P_{A^c}(M_1)P_{A^c}(M_2)} \geq \frac{P_A(M_1M_2)}{P_A(M_1)P_A(M_2)}, \tag{1.2}$$

where all denominators are positive.

Remark. If equality holds in (1.2), then two events M_1 and M_2 are equally informationed conditionally under A^c and A (see Barlow and Proschan(1981)).

Example 2. Consider throwing two fair die.

Let $M_1 = \{(x,y) : y = 3,4,5\}, M_2 = \{(x,y) : x = 1,2\}, M_3 = M_2 \bigcup \{(3,3)\}$, and $A = \{(x,y) : x + y = 7\}$.

Then, (1) M_1 and M_3 are conditionally independent with respect to A, but not mutually inependent. M_1 carries positive information about M_3.

(2) M_1 and M_3 are more strongly informationed conditionally given A^c than given A.

since $\frac{P_{A^c}(M_1M_3)}{P_{A^c}(M_1)P_{A^c}(M_3)} = \frac{12}{11}$, $\frac{P_A(M_1M_3)}{P_A(M_1)P_A(M_3)} = 1$.
(3) M_1 and M_2 are equally informationed conditionally under A^c and A,
since $\frac{P_{A^c}(M_1M_2)}{P_{A^c}(M_1)P_{A^c}(M_2)} = \frac{P_A(M_1M_2)}{P_A(M_1)P_A(M_2)} = 1$.

Definition 3. Events $M_1, M_2, ..., M_k$ are more strongly informationed conditionally given A^c than given A, if

$$\frac{P_{A^c}(\bigcap_{i=1}^k M_i)}{\prod_{i=1}^k P_{A^c}(M_i)} \geq \frac{P_A(\bigcap_{i=1}^k M_i)}{\prod_{i=1}^k P_A(M_i)}, \text{ if all denominators are positive.} \quad (1.3)$$

We obtain equally informationed conditionally when the equality in (1.3) holds(see Barlow and Proschan(1981))

Theorem 3. If events $M_1, M_2, ..., M_k$ are more strongly informationed conditionally given A^c than given A, and $P(\bigcap_{i=1}^k M_i) > 0$, then the odds ratio of an event $A(0 < P(A) < 1)$ has a lower bound and an upper bound:

$$P(A|\bigcap_{i=1}^k M_i) \cdot \prod_{i=1}^k \frac{P(M_i|A^c)}{P(M_i|A)} < \frac{P(A)}{P(A^c)} < \frac{1}{P(A^c|\bigcap_{i=1}^k M_i)} \cdot \prod_{i=1}^k \frac{P(M_i|A^c)}{P(M_i|A)},$$

if all denominators are positive.
Proof. From Bayes' formula of the conditional probability(Rohatgi(1976)), we obtain

$$P(A|\bigcap_{i=1}^k M_i) = \frac{P(A)P(\bigcap_{i=1}^k M_i|A)}{P(A)P(\bigcap_{i=1}^k M_i|A) + P(A^c)P(\bigcap_{i=1}^k M_i|A^c)}$$

By the same proof method of Theorem 1,

$$P(A|\bigcap_{i=1}^k M_i) < \frac{P(A)P(\bigcap_{i=1}^k M_i|A)}{P(A^c)P(\bigcap_{i=1}^k M_i|A^c)}.$$

From definition of more strongly informationed conditionally, that is ,(2)-inequality, $\frac{P(\bigcap_{i=1}^k M_i|A)}{P(\bigcap_{i=1}^k M_i|A^c)} \leq \frac{\prod_{i=1}^k P(M_i|A)}{\prod_{i=1}^k P(M_i|A^c)}$, and so we can obtain the lower bound.
Simiarly, from the same proofing method of last part in Theorem 1, we can obtain the upper bound of the odds ratio of an event A.

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