A Characterization on Strong Reducibility of Near-Rings

Cho, Yong Uk (Silla University)

We shall introduce new concepts of near-rings, that is, strong reducibility and left semi $\pi$-regular near-rings. We will study every strong reducibility of near-ring implies reducibility of near-ring but this converse is not true, and also some characterizations of strong reducibility of near-rings. We shall investigate some relations between strongly reduced near-rings and left strongly regular near-rings, and apply strong reducibility of near-rings to the study of left semi $\pi$-regular near-rings, s-weekly regular near-rings and some other regularity of near-rings.

1. Introduction

A near-ring is a set $N$ with two binary operations $+$ and $\cdot$ such that $(N, +)$ is a not necessarily abelian group with identity $0$, $(N, \cdot)$ is a semigroup and $(a + b)c = ac + bc$ for all $a, b, c$ in $N$. In general the extra axiom $a0 = 0$ for all $a$ in $N$ is said to give a zero symmetric near-ring. Let $(G, \cdot)$ be a group (not necessarily abelian).

If we let $f, g \in M(G) := \{f \mid f : G \to G\}$ and define the sum $f + g$ of the two mappings in $M(G)$ by the rule $(f + g)(x) = f(x) + g(x)$ for all $x \in G$ and the product $f \cdot g$ by the rule $(f \cdot g)(x) = f(g(x))$ for all $x \in G$, then $(M(G), +, \cdot)$ forms a near-ring. Let $M_0(G) := \{f \in M(G) \mid f(0) = 0\}$. Then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring. For the remainder results and definitions in near-rings, we refer to G. Pilz.

In 1980, G. Mason introduced the notions of left strong regularity, right strong regularity, left regularity and right regularity. He proved that for a zero symmetric unital near-ring, the notions of left strong regularity, left regularity and right regularity are equivalent. Also in 1998, G. Mason researched several properties on strong forms of regularity for near-rings. In 1984, the properties of strong
regularity have been slightly improved by Y. V. Reddy and C. V. L. N. Murty, and also in 1986, that of strong regularity and strong \( \pi \)-regularity of semigroup were investigated by M. Hongan.

\( N \) is said to be \textit{left strongly regular} if for all \( a \in N \) there exists \( x \in N \) such that \( a = xa^2 \), that is, \( a \) is a left strongly regular element and \( N \) is left regular if \( N \) is left strongly regular and regular. Right strong regularity and right regularity are defined in a symmetric way. In the ring theory, left strong regularity and right strong regularity are equivalent, but in near-ring theory, they are different in G. Mason. So we say that left strongly regular and right strongly regular near-ring is strongly regular.

More generally, a near-ring \( N \) is called \textit{left strongly} \( \pi \)-\textit{regular} if for all \( a \in N \), there exists a positive integer \( n \) such that \( a^n \) is a left strongly regular element, \textit{left} \( \pi \)-\textit{regular} if it is left strongly \( \pi \)-regular and \( \pi \)-regular. In a symmetry way, \textit{right} strong \( \pi \)-regularity and right \( \pi \)-regularity are defined. Also, a left strongly \( \pi \)-regular and right strongly \( \pi \)-regular near-ring is called \textit{strongly} \( \pi \)-\textit{regular}.

A near-ring \( N \) is called \textit{left s-unital} if \( a \in Na \), \textit{reduced} if it has no nonzero nilpotent element, \textit{left bipotent} if \( Na = Na^2 \) for all \( a \in N \), and \textit{right bipotent} if \( aN = a^2N \) for all \( a \in N \) [1]. The definitions of regularity and \( \pi \)-regularity for near-rings are the same concepts as for rings.

2. Some Characterizations of Strongly Reduced Near-Rings

Recall that a near-ring \( N \) is reduced if, for \( a \in N \), \( a^2 = 0 \) implies \( a = 0 \). In we can find some properties of zero-symmetric reduced near-rings. For a near-ring \( N \), \( N_c \) denotes the constant part of \( N \), that is, \( N_c = \{ x \in N \mid x = x_0 \} \). A near-ring \( N \) is said to be \textit{strongly reduced} if for \( a \in N \), \( a^2 \in N_c \) implies \( a \in N_c \). Obviously \( N \) is strongly reduced if and only if, for \( a \in N \) and any positive integer \( n \), \( a^n \in N_c \) implies \( a \in N_c \). We will show that a strongly reduced near-ring is reduced. A near-ring \( N \) is said to be \textit{left strongly regular} if, for each \( a \in N \), there
exists \( x \in N \) such that \( a = xa^2 \). Right strong regularity is defined in a symmetric way. A two-sided \( N \)-subgroup of \( N \) is a subset \( H \) of \( N \) such that

(i) \((H, +)\) is a subgroup of \((N, +)\),
(ii) \( NH \subseteq H \),
(iii) \( HN \subseteq H \)

For a subset \( S \) of a near-ring \( N \), \( \langle S \rangle \) denotes the two-sided \( N \)-subgroup of \( N \) generated by \( S \).

We give some properties of left strongly regular near-rings.

**Lemma 2.1.** Let \( N \) be a zero symmetric and reduced near-ring. Then for any \( a, b \in N \), \( ab = 0 \) implies \( ba = 0 \).

**Lemma 2.2.** Let \( N \) be a left strongly regular near-ring. If \( a = xa^2 \) for some \( a, x \) in \( N \) then \( a = axa \) and \( ax = xa \).

**Theorem 2.3.** Let \( N \) zero symmetric near-ring. Then the following statements are equivalent:

1. \( N \) is left s-unital and left bipotent;
2. \( N \) is reduced and left bipotent;
3. \( N \) is left strongly regular;
4. \( N \) is regular and for any \( a \in N \) there exists \( x \in N \) such that \( ax = xa \);
5. \( N \) is left regular

**Proof.** (1)\( \Rightarrow \)(2). Suppose \( N \) is left s-unital and left bipotent. Let \( a \) be a nilpotent element in \( N \). Then there exists a positive integer \( n \) such that \( a^n = 0 \). Since \( N \) is left s-unital and left bipotent,
\[ a \in Na = Na^2 = Na^3 = \cdots = Na^n = N0 = \{0\}. \]

Hence \( N \) is reduced.

(2)\( \Rightarrow \)(3). Assume \( N \) is reduced and left bipotent. Then for each \( a \in N \),
\[ Na = Na^2 = Na^3. \]

So we have the equation \( a^2 = xa^3 \) for some \( x \) in \( N \). This implies that 
\[(a - xa^2)a = 0. \] By Lemma 2.1, \( a(a - xa^2) = 0. \) Thus 
\[(a - xa^2)^2 = a(a - xa^2) - xa^2(a - xa^2) = 0 - 0 = 0 \]

Since \( N \) is reduced, \( a = xa^2. \) Hence \( N \) is left strongly regular.

(3)\(\Rightarrow\)(4). This is proved from Lemma 2.2.

(4)\(\Rightarrow\)(1). Suppose the conditions (4). Let \( a \in N. \) Then there exists \( x \in N \) such that \( a = axa \) and \( ax = xa. \) Thus clearly, \( a \in Na \) so that \( N \) is left s-unital. On the other hand, for any \( ra \in Na, \)
\[ ra = raxa = rx a^2 \]
which is contained in \( Na^2. \) Hence \( N \) is left bipotent.

(4)\(\Rightarrow\)(5)\(\Rightarrow\)(3) are easily proved.

We state some examples and basic properties of a strongly reduced near-rings.

**Examples 1.**

1. Let \( N \) be a near-ring. If \( a \in \langle a^2 \rangle \) for each \( a \in N, \) then \( N \) is strongly reduced. In fact, if \( a^2 \in N_c \) then \( a \in \langle a^2 \rangle \subseteq N_c. \) In particular, right or left strongly regular near-rings are strongly reduced.

2. Every integral near-ring \( N \) is strongly reduced. To see this, let \( a \in N \) with \( a^2 \in N_c. \) Then \( (a - a^2)a = 0, \) and hence \( a = a^2 \in N_c. \)

3. Every constant near-ring is strongly reduced.

4. \( Z_2 \) and \( Z_2[x] \) are strongly reduced.

**Proposition 2.4.** Let \( N \) be a strongly reduced near-ring. Then the following statements holds:

1. \( N \) is reduced.

2. For any \( a, b \in N, \) \( ab = 0 \) implies \( ba = b0. \)

*Proof.* (1) Assume that \( a^2 = 0. \) Then \( a^2 \in N_c. \) Hence by hypothesis \( a \in N_c. \) Then
we see \( a = a0 = a0a = aa = 0 \).

(2) Assume that \( ab = 0 \). Then \((ba)^2 = baba = b0 \in N_c \). Hence \( ba \in N_c \). Therefore \((ba)^2 - ba = (ba)^2 - ba = (babab - bab = b0) - b0 = 0\).

Hence we obtain \( ba = (ba)^2 = b0 \).

**Proposition 2.5.** For a strongly reduced near-ring \( N \), we have the following statement:

If \( e \in N \) is an idempotent, then \( eae = ea \) for all \( a \in N \).

**Proof.** Since \((ea - ea)e = 0\), by above Proposition 2.4(2) we have \( e(ea - ea) \in N_c \). Similarly, from \((ea - ea)e = 0\) we obtain \( ea(ea - ea) \in N_c \).

Hence \((ea - ea)^2 = eae(ea - ea) - ea(ea - ea) \in N_c \). Since \( N \) is strongly reduced, \( eae - ea \in N_c \). Therefore \( eae - ea = (ea - ea)e = 0 \).

Clearly, if \( N \) is a zero-symmetric near-ring, then \( N \) is strongly reduced if and only if \( N \) is reduced. The following example shows that a reduced near-ring is not necessarily strongly reduced.

**Examples 2.**

(1) Let \( Z_6 = \{0, 1, 2, 3, 4, 5\} \) with addition modulo 6 and define multiplication as follows:

\[
\begin{array}{c|cccccc}
  & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 3 & 3 & 1 & 3 & 1 & 1 \\
2 & 0 & 0 & 2 & 0 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 0 & 0 & 4 & 0 & 4 & 4 \\
5 & 3 & 3 & 5 & 3 & 5 & 5 \\
\end{array}
\]
Obviously this is a reduced near-ring. The constant part of $\mathbb{Z}_q$ is \{0, 3\}. Since $1^2=3$ is a constant element but 1 is not, this near-ring is not strongly reduced.

(2) Let $N=\{0, 1, 2, 3, 4, 5\}$ be additive group of integers modulo 6 and multiplication as follows:

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Also, this $(N, +, \cdot)$ is a reduced near-ring. The constant part of $N$ is \{0, 2, 4\}. But this near-ring is not strongly reduced, because $1^2=4$ is a constant element but 1 is not a constant element. On the other hand, this near-ring $N$ is an example of $\pi$-regular but not regular.

Now, we obtain some characterizations of strong reducibility of near-rings.

**Theorem 2.6.** The following statements are equivalent for a near-ring $N$:

1. $N$ is strongly reduced.
2. For $a \in N$ and any positive integer $n \geq 2$, $a^n = a^{n-1}$ implies $a^2 = a$.
3. For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.
4. For $a \in N$ and any positive integers $m > n \geq 1$, $a^m = a^n$ implies $a^{m-n+1} = a$.
5. For $a \in N$ and any positive integers $n \geq 1$, $a^n \in N_c$ implies $a \in N_c$.

**Proof.** (1) $\implies$ (2) Assume that $a^n = a^{n-1}$, for $a \in N$ and a positive integer $n \geq 2$. Then $(a^{n-1} - a^{n-2})a = 0$, whence $a(a^{n-1} - a^{n-2}) = a0 \in N_c$ by Proposition 2.4(2).

Since $n \geq 2$, $(a^{n-1} - a^{n-2})^2 = a^{n-2}(a(a^{n-1} - a^{n-2}) - a^{n-3}(a(a^{n-1} - a^{n-2})) \in N_c$. 
This implies $a^{n-1} - a^{n-2} \in N_c$ because $N$ is strongly reduced.  
Hence $a^{n-1} - a^{n-2} = (a^{n-1} - a^{n-2})a = 0$. Continuing this process, we obtain $a^2 = a$.  
(2)⇒(3). Special case of condition (2) for $n = 3$.  
(3)⇒(1). Assume $a^2 \in N_c$. Then $a^3 = a^2a = a^20 = a^2$. By condition (3), this implies $a = a^2 \in N_c$. Hence $N$ is strongly reduced.  
The proof of (1)⇒(4)⇒(5)⇒(1) is left to the readers.  

The following are useful statements.  

**Proposition 2.7.** Let $N$ be a strongly reduced near-ring and $a, b, x \in N$.  
(1) If $ab \in N_c$, then $aNb \subseteq N_c$ and $(ba) \cup bNa \subseteq N_c$.  
(2) If $ab^n \in N_c$ for some positive integer $n$, then $ab \in N_c$.  
(3) If $ab^n = 0$ for some positive integer $n$, then $ab = 0$.  
(4) If $a^{n+1} = xa^{n+1}$ for some positive integer $n$, then $a = xa = ax$.  

**Proof.** (1) since $ab \in N_c$, $(ba)^2 = babab = bab\{a = bab\} \in N_c$. Since $N$ is strongly reduced, we have $bxa \in N_c$. Then we obtain $xb \in N_c$ for each $x \in N$, whence $(axb)^2 \in N_c$. By the strong reducibility of $N$, we obtain $axb \in N_c$ for each $x \in N$. Similarly we can prove that $bNa \subseteq N_c$.  
(2) If $ab^n \in N_c$, then $(ab)^n \in N_c$ by (1). Since $N$ is strongly reduced, this implies $ab \in N_c$.  
(3) If $ab^n = 0$ for some $n \geq 1$, then $ab \in N_c$ by (2). Hence $ab = abb^{n-1} = ab^n = 0$.  
(4) Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. Then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by (3), and so $(a - xa)^2 \in N_c$ by Proposition 2.4(2). Since $N$ is strongly reduced, we have $a - xa \in N_c$. Then $a - xa = (a - xa)a = 0$, that is $a = xa$. Now $(a - xa)a = a^2 - axa = a^2 - a^2 = 0 \in N_c$.  

Hence \((a-ax)62 = a(a-ax) - ax(a-ax) \in N_c\) by (1), and so \(a-ax \in N_c\).

Therefore \(a-ax = (a-ax)a = 0\).

Left strongly regular near-rings are studied by several authors. Since all left strongly regular near-rings are strongly reduced, we can use it to study left strongly regular near-rings. P. Dheena studied s-weakly regular near-rings. We shall show that special s-weakly regular near-rings are left strongly regular.

**Proposition 2.8.** Let \(N\) be a strongly reduced. If for each \(a \in N\), there exists \(x, y \in N\) such that \(a = xa^2ya\), then \(N\) is left strongly regular.

Proof. By hypothesis, \(N\) is strongly reduced. If \(a = xa^2ya\), then \(ya = yxa^2(ya)\). By Proposition 2.7(4). \(ya = yayxa^2\). Thus \(a = xa^2yayxa^2\).

This implies that \(a = xa^2yayxa^2\) \(N\) is left strongly regular.

A near-ring \(N\) is said to be left semi \(\pi\)-regular if, for each \(a \in N\), there exists an element \(x \in N\) such that \(a^n = axa^n\) for some positive integer \(n\). This is a general concept of Von Neumann regularity. The following two theorems are also an application of Proposition 2.7.

**Theorem 2.9.** Let \(N\) be a strongly reduced near-ring. If \(N\) is left semi \(\pi\)-regular, then \(N\) is Von Neumann regular and right strongly regular.

Proof. Suppose \(N\) is a left semi \(\pi\)-regular near-ring. Then for each \(a \in N\), there exists an element \(x \in N\) such that \(a^n = axa^n\) for some positive integer \(n\). Thus we see that \(a^{n+1} = axa^{n+1}\) for some nonnegative integer \(n\). This implies that \(a = axa = a^2x\) by Proposition 2.7(4). Hence \(N\) is Von Neumann regular and right strongly regular.

The following is a generalization of [10, Theorem 3].

**Theorem 2.10.** Let \(N\) be a strongly reduced near-ring and let \(a, x \in N\). If
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\[ a^n = xa^{n+1} \] for some positive integer \( n \), then \( a = xa^2 = axa \) and \( ax = xa \).

**Proof.** Assume that \( a^n = xa^{n+1} \) for some \( n \geq 1 \). By Proposition 2.7 (4), \( a = xa^2 = axa \). Then \( (ax - xa)a = 0 \). Hence, by Proposition 2.7 (1),
\[(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c. \] Since \( N \) is strongly reduced, \( ax - xa \in N_c \). Hence \( ax - xa = (ax - xa)a = 0 \).

A near-ring \( N \) is said to be *left strongly \( \pi \)-regular* if, for each \( a \in N \), there exists a positive integer \( n \) and an element \( x \in N \) such that \( a^n = xa^{n+1} \). As stated in Example 1(1), a right strongly regular near-ring is strongly reduced. Hence the following corollary can be considered as a generalization of [10, Theorem 15].

**Corollary 2.11.** Let \( N \) be a near-ring. Then the following statements are equivalent:

1. \( N \) is a left strongly regular.
2. \( N \) is strongly reduced and left strongly \( \pi \)-regular.

**References**


