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A Characterization on Strong Reducibility of Near-Rings

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We shall introduce new concepts of near-rings, that is, strong reducibility and left semi π-regular near-rings. We will study every strong reducibility of near-ring implies reducibility of near-ring but this converse is not true, and also some characterizations of strong reducibility of near-rings. We shall investigate some relations between strongly reduced near-rings and left strongly regular near-rings, and apply strong reducibility of near-rings to the study of left semi π-regular near-rings, s-weekly regular near-rings and some other regularity of near-rings.

1. Introduction

A *near-ring* is a set N with two binary operations $+$ and $\cdot$ such that $(N, +)$ is a not necessarily abelian group with identity 0, $(N, \cdot)$ is a semigroup and $(a+b)c = ac+bc$ for all a, b, c in N. In general the extra axiom $a0=0$ for all a in N is said to give a *zero symmetric* near-ring. Let $(G, \cdot)$ be a group (not necessarily abelian).

If we let $f, g\, M(G) := \{f \mid f : G \rightarrow G\}$ and define the sum $f+g$ of the two mappings in $M(G)$ by the rule $(f+g)(x) = f(x)+g(x)$ for all $x \in G$ and the product $f \cdot g$ by the rule $(f \cdot g)(x) = f(g(x))$ for all $x \in G$, then $(M(G), +, \cdot)$ forms a near-ring. Let $M_0(G) := \{f \in M(G) \mid f(0)=0\}$. Then $(M_0(G), +, \cdot)$ is a zero symmetric near-ring. For the remainder results and definitions in near-rings, we refer to G. Pilz.

In 1980, G. Mason introduced the notions of left strong regularity, right strong regularity, left regularity and right regularity. He proved that for a zero symmetric unital near-ring, the notions of left strong regularity, left regularity and right regularity are equivalent. Also in 1998, G. Mason researched several properties on strong forms of regularity for near-rings. In 1984, the properties of strong regularity have been slightly improved by Y. V. Reddy and C. V. L. N. Murty, and also in 1986, that of strong regularity and strong π-regularity of semigroup were investigated by M. Hongan.

N is said to be *left strongly regular* if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$, that is, a is a left strongly regular element and N is left regular if N is left strongly regular and regular. Right strong regularity and right regularity are defined in a symmetric way. In the ring theory, left strong regularity and right strong regularity are equivalent, but in near-ring theory, they are different in G. Mason. So we say that left strongly regular and right strongly regular near-ring is strongly regular.

More generally, a near-ring N is called *left strongly π-regular* if for all $a \in N$, there exists a positive integer n such that a^n is a left strongly regular element, *left π-regular* if it is left strongly π-regular and π-regular. In a symmetry way, right strong π-regularity and right π-regularity are defined. Also, a left strongly π-regular and right strongly π-regular near-ring is called *strongly π-regular.*

A near-ring N is called *left s-unital* if $a \in Na$, *reduced* if it has no nonzero nilpotent element, *left bipotent* if $Na = Na^2$ for all $a \in N$, and *right bipotent* if $aN = a^2N$ for all $a \in N$ [1]. The definitions of regularity and π-regularity for near-rings are the same concepts as for rings.

2. Some Characterizations of Strongly Reduced Near-Rings

Recall that a near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$. In we can find some properties of zero-symmetric reduced near-rings. For a near-ring N, N_c denotes the constant part of N, that is, $N_c = \{x \in N \mid x = x0\}$. A near-ring N is said to be *strongly reduced* if for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously N is strongly reduced if and only if, for $a \in N$ and any positive integer n, $a^n \in N_c$ implies $a \in N_c$. We will show that a strongly reduced near-ring is reduced. A near-ring N is said to be *left strongly regular* if, for each $a \in N$, there exists $x \in N$ such that $a = xa^2$. Right strong regularity is defined in a symmetric way. A *two-sided N-subgroup* of N is a subset H of N such that

(i) $(H, +)$ is a subgroup of $(N, +)$,
(ii) $NH \subseteq H$,
(iii) $HN \subseteq H$

For a subset S of a near-ring N, $\langle S \rangle$ denotes the two-sided N-subgroup of N generated by S.
We give some properties of left strongly regular near-rings.

Lemma 2.1. Let N be a zero symmetric and reduced near-ring. Then for any $a, b \in N$, $ab = 0$ implies $ba = 0$.

Lemma 2.2. Let N be a left strongly regular near-ring. If $a = xa^2$ for some a, x in N then $a = axa$ and $ax = xa$.

Theorem 2.3. Let N zero symmetric near-ring. Then the following statements are equivalent :

(1) N is left s-unital and left bipotent;
(2) N is reduced and left bipotent;
(3) N is left strongly regular;
(4) N is regular and for any $a \in N$ there exists $x \in N$ such that $ax = xa$;
(5) N is left regular

Proof. (1)⇒(2). Suppose N is left s-unital and left bipotent. Let a be a nilpotent element in N. Then there exists a positive integer n such that $a^n = 0$. Since N is left s-unital and left bipotent,

$$a \in Na = Na^2 = Na^3 = \cdots = Na^n = N0 = \{0\}.$$

Hence N is reduced.
(2)⇒(3). Assume N is reduced and left bipotent. Then for each $a \in N$,

$$Na = Na^2 = Na^3.$$

So we have the equation $a^2 = xa^3$ for some x in N. This implies that $(a - xa^2)a = 0$. By Lemma 2.1, $a(a - xa^2) = 0$. Thus

$$(a - xa^2)^2 = a(a - xa^2) - xa^2(a - xa^2) = 0 - 0 = 0$$

Since N is reduced, $a = xa^2$. Hence N is left strongly regular.

(3)⇒(4). This is proved from Lemma 2.2.

(4)⇒(1). Suppose the conditions (4). Let $a \in N$. Then there exists $x \in N$ such that $a = axa$ and $ax = xa$. Thus clearly, $a \in Na$ so that N is left s-unital. On the other hand, for any $ra \in Na$,

$$ra = raxa = rxa^2$$

which is contained in Na^2. Hence N is left bipotent.

(4)⇒(5)⇒(3) are easily proved.

We state some examples and basic properties of a strongly reduced near-rings.

Examples 1.

(1) Let N be a near-ring. If $a \in \langle a^2 \rangle$ for each $a \in N$, then N is strongly reduced. In fact, if $a^2 \in N_c$ then $a \in \langle a^2 \rangle \sqsubseteq N_c$. In particular, right or left strongly regular near-rings are strongly reduced.

(2) Every integral near-ring N is strongly reduced. To see this, let $a \in N$ with $a^2 \in N_c$. Then $(a - a^2)a = 0$, and hence $a = a^2 \in N_c$.

(3) Every constant near-ring is strongly reduced.

(4) Z_2 and $Z_2[x]$ are strongly reduced.

Proposition 2.4. Let N be a strongly reduced near-ring. Then the following statements holds :

(1) N is reduced.

(2) For any $a, b \in N$, $ab = 0$ implies $ba = b0$.

Proof. (1) Assume that $a^2 = 0$. Then $a^2 \in N_c$. Hence by hypothesis $a \in N_c$. Then we see $a = a0 = a0a = aa = 0$.

(2) Assume that $ab = 0$. Then $(ba)^2 = baba = b0 \in N_c$. Hence $ba \in N_c$. Therefore $(ba)^2 - ba \in N_c$. Then $(ba)^2 - ba = \{(ba)^2 - ba\}b = babab - bab = b0 - b0 = 0$.

Hence we obtain $ba = (ba)^2 = b0$.

Proposition 2.5. For a strongly reduced near-ring N, we have the following statement :

If $e \in N$ is an idempotent, then $eae = ea$ for all $a \in N$.

Proof. Since $(eae - ea)e = 0$, by above Proposition 2.4(2) we have $e(eae - ea) \in N_c$. Similarly, from $(eae - ea)ea = 0$ we obtain $ea(eae - ea) \in N_c$.

Hence $(eae - ea)^2 = eae(eae - ea) - ea(eae - ea) \in N_c$. Since N is strongly redeced, $eae - ea \in N_c$. Therefore $eae - ea = (eae - ea)e = 0$.

Clearly, if N is a zero-symmetric near-ring, then N is strongly reduced if and only if N is reduced. The following example shows that a reduced near-ring is not necessarily strongly reduced.

Examples 2.

(1) Let $Z_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and define multiplication as follows :

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

Obviously this is a reduced near-ring. The constant part of Z_6 is $\{0,3\}$. Since $1^2=3$ is a constant element but 1 is not, this near-ring is not strongly reduced.

(2) Let $N=\{0,1,2,3,4,5\}$ be additive group of integers modulo 6 and multiplication as follows :

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	4	4	4	1	4	1
2	2	2	2	2	2	2
3	0	0	0	3	0	3
4	4	4	4	4	4	4
5	2	2	2	5	2	5

Also, this $(N,+,\cdot)$ is a reduced near-ring. The constant part of N is $\{0,2,4\}$. But this near-ring is not strongly reduced, because $1^2=4$ is a constant element but 1 is not a constant element. On the other hand, this near-ring N is an example of π-regular but not regular.

Now, we obtain some characterizations of strong reducibility of near-rings.

Theorem 2.6. The following statements are equivalent for a near-ring N :

(1) N is strongly reduced.

(2) For $a\in N$ and any positive integer $n>2$, $a^n=a^{n-1}$ implies $a^2=a$.

(3) For $a\in N$, $a^3=a^2$ implies $a^2=a$;

(4) For $a\in N$ and any positive integers $m>n>1$, $a^m=a^n$ implies $a^{m-n+1}=a$;

(5) For $a\in N$and any positive integers $n>1$, $a^n\in N_c$ implies $a\in N_c$.

Proof. (1)⇒(2) Assume that $a^n=a^{n-1}$, for $a\in N$ and a positive integer $n>2$. Then $(a^{n-1}-a^{n-2})a=0$, whence $a(a^{n-1}-a^{n-2})=a0\in N_c$ by Proposition 2.4(2).

Since $n>2$, $(a^{n-1}-a^{n-2})^2=a^{n-2}\{a(a^{n-1}-a^{n-2})\}-a^{n-3}\{a(a^{n-1}-a^{n-2})\}\in N_c$.

This implies $a^{n-1}-a^{n-2}\in N_c$ because N is strongly reduced.
Hence $a^{n-1}-a^{n-2}=(a^{n-1}-a^{n-2})a=0$. Continuing this process, we obtain $a^2=a$.
(2)⇒(3). Special case of condition (2) for $n=3$.
(3)⇒(1). Assume $a^2\in N_c$. Then $a^3=a^2a=a^20=a^2$. By condition (3), this implies $a=a^2\in N_c$. Hence N is strongly reduced.
The proof of (1)⇒(4)⇒(5)⇒(1) is left to the readers.

The following are useful statements.

Proposition 2.7. Let N be a strongly reduced near-ring and $a, b, x\in N$.
(1) If $ab\in N_c$, then $aNb \subseteq N_c$ and $\{ba\}\bigcup bNa \subseteq N_c$.
(2) If $ab^n\in N_c$ for some positive integer n, then $ab\in N_c$.
(3) If $ab^n=0$ for some positive integer n, then $ab=0$
(4) If $a^{n+1}=xa^{n+1}$ for some positive integer n, then $a=xa=ax$.

Proof. (1) since $ab\in N_c$, $(ba)^2=baba=bab0a=bab0\in N_c$. Since N is strongly reduced, we have $ba\in N_c$. Then we obtain $xba\in N_c$ for each $x\in N$, whence $(axb)^2\in N_c$. By the strong reducibility of N, we obtain $axb\in N_c$ for each $x\in N$. Similarly we can prove that $bNa \subseteq N_c$.
(2) If $ab^n\in N_c$, then $(ab)^n\in N_c$ by (1). Since N is strongly reduced, this implies $ab\in N_c$.
(3) If $ab^n=0$ for some $n\geq 1$, then $ab\in N_c$ by (2). Hence $ab=abb^{n-1}=ab^n=0$.
(4) Suppose $a^{n+1}=xa^{n+1}$ for some $n\geq 0$. Then $(a-xa)a^n=0$. Hence $(a-xa)a=0$ by (3), and so $(a-xa)^2\in N_c$ by Proposition 2.4(2). Since N is strongly reduced, we have $a-xa\in N_c$. Then $a-xa=(a-xa)a=0$, that is $a=xa$. Now $(a-ax)a=a^2-axa=a^2-a^2=0\in N_c$.

Hence $(a-ax)62=a(a-ax)-ax(a-ax)\in N_c$ by (1), and so $a-ax\in N_c$. Therefore $a-ax=(a-ax)a=0$.

Left strongly regular near-rings are studied by several authors. Since all left strongly regular near-rings are strongly reduced, we can use it to study left strongly regular near-rings. P. Dheena studied s-weakly regular near-rings. We shall show that special s-weakly regular near-rings are left strongly regular.

Proposition 2.8. Let N be a strongly reduced. If for each $a\in N$, there exists $x, y\in N$ such that $a=xa^2ya$, then N is left strongly regular.

Proof. By hypothesis, N is strongly reduced. If $a=xa^2ya$, then $ya=yxa^2(ya)$. By Proposition 2.7(4). $ya=yayxa^2$. Thus $a=xa^2yayxa^2$.

This implies that $a=xa^2yayxa^2$ N is left strongly regular.

A near-ring N is said to be *left semi π-regular* if, for each $a\in N$, there exists an element $x\in N$ such that $a^n=axa^n$ for some positive integer n. This is a general concept of Von Neumann regularity. The following two theorems are also an application of Proposition 2.7.

Theorem 2.9. Let N be a strongly reduced near-ring. If N is left semi π -regular, then N is Von Neumann regular and right strongly regular.

Proof. Suppose N is a left semi π-regular near-ring. Then for each $a\in N$, there exists an element $x\in N$ such that $a^n=axa^n$ for some positive integer n. Thus we see that $a^{n+1}=axa^{n+1}$ for some nonnegative integer n. This implies that $a=axa=a^2x$ by Proposition 2.7(4). Hence N is Von Neumann regular and right strongly regular.

The following is a generalization of [10, Theorem 3].

Theorem 2.10. Let N be a strongly reduced near-ring and let $a, x\in N$. If

$a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.

Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By Proposition 2.7 (4), $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by Proposition 2.7 (1), $(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c$. Since N is strongly reduced, $ax - xa \in N_c$. Hence $ax - xa = (ax - xa)a = 0$.

A near-ring N is said to be *left strongly π-regular* if, for each $a \in N$, there exists a positive integer n. and an element $x \in N$ such that $a^n = xa^{n+1}$. As stated in Example 1(1). a right strongly regular near-ring is strongly reduced. Hence the following corollary can be considered as a generalization of [10, Theorem 15].

Corollary 2.11. Let N be a near-ring. Then the following statements are equivalent :

(1) N is a left strongly regular.

(2) N is strongly reduced and left strongly π-regular.

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