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# TENSOR PRODUCTS OF $C^*$–ALGEBRAS WITH FIBRES GENERALIZED NONCOMMUTATIVE TORI AND CUNTZ ALGEBRAS

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Abstract. The generalized noncommutative torus $T_\rho^d$ of rank $m$ was defined in [2]. Assume that for the completely irrational noncommutative subtorus $A_\rho$ of rank $m$ of $T_\rho^d$ there is no integer $q > 1$ such that $\operatorname{tr}(K_0(A_\rho)) = \frac{1}{q} \cdot \operatorname{tr}(K_0(A_{\rho'}))$ for $A_{\rho'}$ a completely irrational noncommutative torus of rank $m$. All $C^*$-algebras $\Gamma(\eta)$ of sections of locally trivial $C^*$-algebra bundles $\eta$ over $M = \prod_{i=1}^{e} S^{2k_i} \times \prod_{i=1}^{s} S^{2n_i+1}$, $\prod_{i=1}^{s} \mathbb{PR}_{2n_i}$, or $\prod_{i=1}^{s} L_{k_i}(n_i)$ with fibres $T_\rho^d \otimes M_c(\mathbb{C})$ were constructed in [6, 7, 8]. We prove that $\Gamma(\eta) \otimes M_{p^\infty}$ is isomorphic to $C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of $cd$ is a subset of the set of prime factors of $p$, that $\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if $cd$ and $2u-1$ are relatively prime, and that $\mathcal{O}_\infty \otimes \Gamma(\eta)$ is not isomorphic to $\mathcal{O}_\infty \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$ when no non-trivial matrix algebra can be factored out of $\Gamma(\eta)$.

## 0. Introduction

Given a locally compact abelian group $G$ and a multiplier $\omega$ on $G$, one can associate to them the twisted group $C^*$-algebra $C^*(G, \omega)$. $C^*(\mathbb{Z}^m, \omega)$ is said to be a *noncommutative torus of rank $m$* and denoted by $A_\omega$. The multiplier $\omega$ determines a subgroup $S_\omega$ of $G$, called its *symmetry group*, and the multiplier $\omega$ is called *totally skew* if the symmetry group $S_\omega$ is trivial. And $A_\omega$ is called *completely irrational* if $\omega$ is totally skew (see [1]). It was shown in [1] that if $G$ is a locally

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compact abelian group and $\omega$ is a totally skew multiplier on $G$, then $C^*(G,\omega)$ is a simple $C^*$-algebra.

Under the assumption that there is no integer $q > 1$ such that $\mathrm{tr}(K_0(A_\rho)) = \frac{1}{q} \cdot \mathrm{tr}(K_0(A_{\rho'}))$ for $A_{\rho'}$ a completely irrational noncommutative torus of rank $m$, all $C^*$-algebras $\Gamma(\eta)$ of sections of locally trivial $C^*$-algebra bundles $\eta$ over $M = \prod_{i=1}^{e} S^{2k_i} \times \prod_{i=1}^{s} S^{2n_i+1}$, $\prod_{i=1}^{s} \mathbb{PR}_{2n_i}$, or $\prod_{i=1}^{s} L_{k_i}(n_i)$ with fibres $T_\rho^d \otimes M_c(\mathbb{C})$ were constructed in [6, 7, 8]. It was shown in [6, 7, 8] that $[1_{\Gamma(\eta)}] \in K_0(\Gamma(\eta))$ is primitive when no non-trivial matrix algebra can be factored out of $\Gamma(\eta)$.

In this paper, we prove that for each $C^*$-algebra $\Gamma(\eta)$ of which no non-trivial matrix algebra can be factored out, $\Gamma(\eta) \otimes M_{p^\infty}$ is isomorphic to $C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of $cd$ is a subset of the set of prime factors of $p$, that $\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if $cd$ and $2u-1$ are relatively prime, and that $\mathcal{O}_\infty \otimes \Gamma(\eta)$ is not isomorphic to $\mathcal{O}_\infty \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$, where $\mathcal{O}_u$ and $\mathcal{O}_\infty$ denote the Cuntz algebras and the generalized Cuntz algebra, respectively.

## 1. Tensor products of $C^*$-algebras over $CW$-complexes with fibres generalized noncommutative tori and Cuntz algebras

Let $\Gamma(\eta)$ be a $C^*$-algebra of sections of a locally trivial $C^*$-algebra bundle over the manifold $M = \prod_{i=1}^{e} S^{2k_i} \times \prod_{i=1}^{s} S^{2n_i+1}, \prod_{i=1}^{s} \mathbb{PR}_{2n_i}$, or $\prod_{i=1}^{s} L_{k_i}(n_i)$ with fibres $T_\rho^d \otimes M_c(\mathbb{C})$ of which no non-trivial matrix algebra can be factored out. Using the fact that $[1_{\Gamma(\eta)}] \in K_0(\Gamma(\eta))$ is primitive (see [6, 7, 8]), we investigate the bundle structure of $\Gamma(\eta) \otimes M_{p^\infty}$ for $M_{p^\infty}$ a $UHF$-algebra of type $p^\infty$.

1 Theorem. *$\Gamma(\eta) \otimes M_{p^\infty}$ is isomorphic to $C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$ if and only if the set of prime factors of $cd$ is a subset of the set of prime factors of $p$.*

*Proof.* Assume that the set of prime factors of $cd$ is a subset of the set of prime factors of $p$. To show that $\Gamma(\eta) \otimes M_{p^\infty}$ is isomorphic to $C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$, it is enough to show that $\Gamma(\eta) \otimes M_{(cd)^\infty} \cong C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^\infty}$. But there exist the $C^*$-algebra homomorphisms which are the canonical inclusions $\Gamma(\eta) \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{(cd)^g}(\mathbb{C})$ and the $C(M) \otimes A_\rho$-module maps $C(M) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \hookrightarrow \Gamma(\eta) \otimes M_{(cd)^g}(\mathbb{C})$:

$$\begin{aligned} \Gamma(\eta) \hookrightarrow C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \hookrightarrow \Gamma(\eta) \otimes M_{cd}(\mathbb{C}) \\ \hookrightarrow C(M) \otimes A_\rho \otimes M_{(cd)^2}(\mathbb{C}) \hookrightarrow \cdots . \end{aligned}$$

The inductive limit of the odd terms

$$\cdots \to \Gamma(\eta) \otimes M_{(cd)^g}(\mathbb{C}) \to \Gamma(\eta) \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots$$

is $\Gamma(\eta) \otimes M_{(cd)^\infty}$, and the inductive limit of the even terms

$$\cdots \to C(M) \otimes A_\rho \otimes M_{(cd)^g}(\mathbb{C}) \to C(M) \otimes A_\rho \otimes M_{(cd)^{g+1}}(\mathbb{C}) \to \cdots$$

is $C(M) \otimes A_\rho \otimes M_{(cd)^\infty}$. Thus by the Elliott theorem [5, Theorem 2.1], $\Gamma(\eta) \otimes M_{(cd)^\infty}$ is isomorphic to $C(M) \otimes A_\rho \otimes M_{(cd)^\infty}$.

Conversely, assume that $\Gamma(\eta) \otimes M_{p^\infty} \cong C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C}) \otimes M_{p^\infty}$. Then the unit $1_{\Gamma(\eta)} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(M)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}$. So

$$\begin{aligned} [1_{\Gamma(\eta)} \otimes 1_{M_{p^\infty}}] &= [1_{C(M)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] \\ [1_{\Gamma(\eta)} \otimes 1_{M_{p^\infty}}] &= [1_{\Gamma(\eta)}] \otimes [1_{M_{p^\infty}}] \\ [1_{C(M)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}] &= cd([1_{C(M)\otimes A_\rho}] \otimes [1_{M_{p^\infty}}]). \end{aligned}$$

Under the assumption that the unit $1_{\Gamma(\eta)} \otimes 1_{M_{p^\infty}}$ maps to the unit $1_{C(M)\otimes A_\rho} \otimes 1_{M_{p^\infty}} \otimes I_{cd}$, if there is a prime factor $q$ of $cd$ such that $q \nmid p$, then $[1_{M_{p^\infty}}] \neq q[e_\infty]$ for $e_\infty$ a projection in $M_{p^\infty}$. So there is a projection $e \in \Gamma(\eta)$ such that $[1_{\Gamma(\eta)}] = q[e]$. This contradicts the fact that $[1_{\Gamma(\eta)}]$ is primitive. Thus the set of prime factors of $cd$ is a subset of the set of prime factors of $p$.

Therefore, $\Gamma(\eta)\otimes M_{p^\infty}$ is isomorphic to $C(M)\otimes A_\rho\otimes M_{cd}(\mathbb{C})\otimes M_{p^\infty}$ if and only if the set of prime factors of $cd$ is a subset of the set of prime factors of $p$. $\square$

Now let us study the bundle structure of the tensor products of $\Gamma(\eta)$ with (even) Cuntz algebras.

The Cuntz algebra $\mathcal{O}_u, 2 \leq u < \infty$, is the universal $C^*$-algebra generated by $u$ isometries $s_1, \ldots, s_u$, i.e., $s_j^* s_j = 1$ for all $j$, with the relation $s_1 s_1^* + \cdots + s_u s_u^* = 1$. Cuntz [3, 4] proved that $\mathcal{O}_u$ is simple and the $K$-theory of $\mathcal{O}_u$ is $K_0(\mathcal{O}_u) = \mathbb{Z}/(u-1)\mathbb{Z}$ and $K_1(\mathcal{O}_u) = 0$. He proved that $K_0(\mathcal{O}_u)$ is generated by the class of the unit.

2 Proposition. *Let $u$ be a positive integer such that $cd$ and $u-1$ are not relatively prime. Then $\mathcal{O}_u \otimes \Gamma(\eta)$ is not isomorphic to $\mathcal{O}_u \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.*

*Proof.* Let $p$ be a prime such that $p \mid cd$ and $p \mid u-1$. Suppose that $\mathcal{O}_u \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_u \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Then the unit $1_{\mathcal{O}_u\otimes\Gamma(\eta)}$ maps to the unit $1_{\mathcal{O}_u\otimes C(M)\otimes A_\rho} \otimes I_{cd}$. So

$$[1_{\mathcal{O}_u\otimes\Gamma(\eta)}] = [1_{\mathcal{O}_u\otimes C(M)\otimes A_\rho} \otimes I_{cd}] = cd[1_{\mathcal{O}_u\otimes C(M)\otimes A_\rho}].$$

Hence there is a projection $e$ in $\mathcal{O}_u \otimes \Gamma(\eta)$ such that $[1_{\mathcal{O}_u\otimes\Gamma(\eta)}] = cd[e]$. But $[1_{\mathcal{O}_u\otimes\Gamma(\eta)}] = [1_{\mathcal{O}_u}] \otimes [1_{\Gamma(\eta)}]$ and $[1_{\mathcal{O}_u}]$ is a generator of $K_0(\mathcal{O}_u) \cong \mathbb{Z}/(u-1)\mathbb{Z}$ (see [4]). But $p \mid u-1$. $[1_{\mathcal{O}_u}] \neq p[e_*]$ for $e_*$ a projection in $\mathcal{O}_u$. So $[1_{\Gamma(\eta)}] = p[e']$ for $e'$ a projection in $\Gamma(\eta)$. This contradicts the fact that $[1_{\Gamma(\eta)}]$ is primitive. Hence $cd$ and $u-1$ are relatively prime.

Therefore, $\mathcal{O}_u\otimes\Gamma(\eta)$ is not isomorphic to $\mathcal{O}_u\otimes C(M)\otimes A_\rho\otimes M_{cd}(\mathbb{C})$ if $cd$ and $u-1$ are not relatively prime. $\square$

The following result is useful to understand the bundle structure of $\mathcal{O}_u \otimes \Gamma(\eta)$.

3 PROPOSITION. ([9, Theorem 7.2]) *Let $A$ and $B$ be unital simple inductive limits of even Cuntz algebras. If $\alpha : K_0(A) \to K_0(B)$ is an isomorphism of abelian groups satisfying $\alpha([1_A]) = [1_B]$, then there is an isomorphism $\phi : A \to B$ which induces $\alpha$.*

4 COROLLARY.

(1) *Let $p$ be an odd integer such that $p$ and $2u-1$ are relatively prime. Then $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{(2u-1)p+1} \otimes M_{p^\infty}$. That is, $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$.*

(2) *$\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty}$.*

5 THEOREM. *$\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if $cd$ and $2u-1$ are relatively prime.*

*Proof.* Assume that $cd$ and $2u-1$ are relatively prime. Let $cd = p2^v$ for some odd integer $p$. Then $p$ and $2u-1$ are relatively prime. Then by Corollary 4 $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty}$, and $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(2u)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2u)^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(2^v)^\infty}$. So $\mathcal{O}_{2u}$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{p^\infty} \otimes M_{(2^v)^\infty} \cong \mathcal{O}_{2u} \otimes M_{(cd)^\infty}$. Thus by Theorem 1 $\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes \Gamma(\eta)$, which in turn is isomorphic to $\mathcal{O}_{2u} \otimes M_{(cd)^\infty} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. Thus $\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$.

The converse was proved in Proposition 2.

Therefore, $\mathcal{O}_{2u} \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_{2u} \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if and only if $cd$ and $2u-1$ are relatively prime. □

Cuntz [4] computed the $K$-theory of the generalized Cuntz algebra $\mathcal{O}_\infty$, generated by a sequence of isometries with mutually orthogonal ranges, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. He proved that $K_0(\mathcal{O}_\infty)$ is generated by the class of the unit.

6 PROPOSITION. *$\mathcal{O}_\infty \otimes \Gamma(\eta)$ is not isomorphic to $\mathcal{O}_\infty \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$ if $cd > 1$.*

*Proof.* Suppose $\mathcal{O}_\infty \otimes \Gamma(\eta)$ is isomorphic to $\mathcal{O}_\infty \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. The unit $1_{\mathcal{O}_\infty \otimes \Gamma(\eta)}$ maps to the unit $1_{\mathcal{O}_\infty \otimes C(M) \otimes A_\rho} \otimes I_{cd}$. By the same trick as in the proof of Proposition 2, one can show that $[1_{\mathcal{O}_\infty \otimes \Gamma(\eta)}] = cd[e]$ for a projection $e \in \mathcal{O}_\infty \otimes \Gamma(\eta)$. $[1_{\mathcal{O}_\infty \otimes \Gamma(\eta)}] = [1_{\mathcal{O}_\infty}] \otimes [1_{\Gamma(\eta)}]$ and $[1_{\mathcal{O}_\infty}]$ is a primitive element of $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$ (see [4]). So $[1_{\Gamma(\eta)}] = cd[e']$ for a projection $e' \in \Gamma(\eta)$. This contradicts the fact that $[1_{\Gamma(\eta)}]$ is primitive.

Therefore, $\mathcal{O}_\infty \otimes \Gamma(\eta)$ is not isomorphic to $\mathcal{O}_\infty \otimes C(M) \otimes A_\rho \otimes M_{cd}(\mathbb{C})$. □

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