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ON BOUNDED OPERATOR Q_q IN WEIGHT BLOCH SPACE

Ki Seong Choi

Abstract. Let D be the open unit disk in the complex plane $\mathbb{C}$. For any $q > 0$, the operator Q_q defined by

$$Q_q f(z) = q \int_D \frac{f(w)}{(1 - z\overline{w})^{1+q}} dw, \ z \in D.$$

maps $L^\infty(D)$ boundedly onto B_q for each $q > 0$. In this paper, weighted Bloch spaces $\mathcal{B}_q \quad (q > 0)$ are considered on the open unit ball in $\mathbb{C}^n$. In particular, we will investigate the possibility of extension of this operator to the Weighted Bloch spaces $\mathcal{B}_q$ in $\mathbb{C}^n$.

1. Introduction

Let D be the open unit disk in the complex plane $\mathbb{C}$. The Bloch space of D consists of holomorphic functions f on D such that

$$\sup\{(1 - |z|^2)|f'(z)| \mid z \in D\} < +\infty$$

Specifically, for each $q > 0$, we let $\mathbb{B}_q$ denote the space of analytic functions f on D satisfying

$$\sup\{(1 - |z|^2)^q |f'(z)| : z \in D\} < +\infty$$

These spaces are a certain type of Besov space.

Throughout this paper, $\mathbb{C}^n$ will be the Cartesian product of n copies of $\mathbb{C}$. For $z = (z_1, z_2, \ldots, z_n)$ and $w = (w_1, w_2, \ldots, w_n)$ in

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$\mathbb{C}^n$, the inner product is defined by $< z, w >= \sum_{j=1}^{n} z_j \overline{w_j}$ and the norm by $\| z \|^2 =< z, z >$.

Let B be the open unit ball in the complex space $\mathbb{C}^n$ and S the boundary of B. The Bergman metric (on B) $b_B : B \times \mathbb{C}^n \longrightarrow \mathbb{R}$ is given by

$$b_B^2(z, \xi) = \frac{n+1}{(1 - \| z \|^2)^2} [\, (1 - \| z \|^2) \| \xi \|^2 + | < z, \xi > |^2 \,] \, .$$

If $f \in H(B)$, where $H(B)$ is the set of holomorphic functions on B, then the quantity Qf is defined by

$$Qf(z) = \sup_{\|\xi\|=1} \frac{|\nabla f(z) \cdot \xi|}{b_B(z, \xi)}, \quad z \in B, \quad \xi \in \mathbb{C}^n \ ,$$

where $\nabla f(z) = (\frac{\partial f}{\partial z_1}, \quad , \frac{\partial f}{\partial z_n})$ is the holomorphic gradient of f.

A holomorphic function $f : B \to \mathbb{C}$ is called a Bloch function if

$$\sup_{z \in B} Qf(z) < \infty \ .$$

In [7], Timoney showed that the linear space of all holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \| z \|^2) \, \| \nabla f(z) \| < \infty$$

is equivalent to the space $\mathcal{B}$ of Bloch functions on B.

In [5] , we have introduced the weighted Bloch Spaces $\mathcal{B}_q (q > 0)$ on the open unit ball B in $\mathbb{C}^n$ which extend the notion of Bloch space $\mathcal{B}$ to larger classes of holomorphic functions on B.

For each $q > 0$, the weighted Bloch space of B, denoted by $\mathcal{B}_q$, consists of holomorphic functions $f : B \to \mathbb{C}$ which satisfy

$$\sup_{z \in B} (1 - \| z \|^2)^q \, \| \nabla f(z) \| < \infty \ .$$

Clearly, $\mathcal{B}_q$ are increasing function spaces of $q > 0$. In particular, $\mathcal{B}_1 = \mathcal{B}$.
We proved in [4] that the space $\mathcal{B}_q$ is a Banach space and that $\mathcal{B}_q$ can be identified with the space of holomorphic functions f with the conditions:

$$\sup\{(1- \parallel z \parallel^2)^{q-1}|f(z)| \mid \quad z \in B\} < \infty$$

for all $q > 0$.

For any $q > 0$, the operator Q_q is defined by

$$Q_q f(z) = q\int_D \frac{f(w)}{(1-z\overline{w})^{1+q}}dw, \ z \in D.$$

It is well known that the operator Q_q maps $L^\infty(D)$ boundedly onto $\mathbb{B}_q$ for each $q > 0$. In this paper, we will investigate the possibility of extension of this operator to the Weighted Bloch spaces $\mathcal{B}_q$ in $\mathbb{C}^n$. For any $q > 0$, let Q_q denote the operator defined by

$$Q_q f(z) = c_{q-1}\int_B \frac{f(w)}{(1- < z, w >)^{n+q}}d\nu(w), \quad z \in B,$$

In this paper, we will prove that Q_q maps $L^\infty(B)$ boundedly onto $\mathcal{B}_q$ and that, for each $q > 0$, the operator Q_q maps $C_0(B)$ onto $\mathcal{B}_q$

2. Weighted Bloch spaces

Let ν be the Lebesgue measure in $\mathbb{C}^n$ normalized by $\nu(B) = 1$. Let σ be the rotation invariant surface measure on S normalized by $\sigma(S) = 1$. The measure μ_q is the weighted Lebesgue measure:

$$d\mu_q = c_q(1- \parallel z \parallel^2)^q d\nu(z),$$

where $q > -1$ is fixed, and c_q is a normalization constant such that $\mu_q(B) = 1$.

THEOREM 1. *If* $f \in L^1_{\mu_q}(B) \cap H(B), \quad q > -1$, *then*

$$f(z) = c_q \int_B \frac{(1- \| w \|^2)^q}{(1- < z, w >)^{n+q+1}} f(w) d\nu(w).$$

Proof. See[5. Theorem 2]. □

THEOREM 2. *Suppose* $q > -1, z \in B$, *and* $f \in \mathcal{B}_q$. *Then*

$$f(z) = f(0) + \frac{c_q}{n+q} \int_B \frac{(1- \| w \|^2)^q \nabla f(w) \cdot z}{< z, w > (1- < z, w >)^{n+q}} d\nu(w).$$

Proof. See[5. Theorem 3]. □

THEOREM 3. *For* $z \in B$, *c is real,* $t > -1$, *define*

$$I_{c,t}(z) = \int_B \frac{(1- \| w \|^2)^t}{|1- < z, w > |^{n+1+c+t}} d\nu(w), \quad z \in B.$$

Then,

(i) $I_{c,t}(z)$ *is bounded in* B *if* $c < 0$;
(ii) $I_{0,t}(z) \sim -\log(1- \| z \|^2)$ *as* $\| z \| \to 1^-$;
(iii) $I_{c,t}(z) \sim (1- \| z \|^2)^{-c}$ *as* $\| z \| \to 1^-$ *if* $c > 0$.

Proof. See [6, Proposition 1.4.10]. □

THEOREM 4. *Suppose* $q > 0$. *Then* f *is in* $\mathcal{B}_q$ *if and only if* $(1- \| z \|^2)^{q-1} |f(z)|$ *is bounded on* B.

Proof. First assume that f is in $\mathcal{B}_q$. By Theorem 2,

$$f(z) = f(0) + \frac{c_q}{n+q} \int_B \frac{(1- \| w \|^2)^q \nabla f(w) \cdot z}{< z, w > (1- < z, w >)^{n+q}} d\nu(w).$$

It follows that

$$|f(z) - f(0)| \le \frac{c_q}{n+q} \| f \|_q \int_B \frac{\| z \|}{| < z, w > ||1- < z, w > |^{n+q}} d\nu(w).$$

The factor $| < z, w > |$ in the denominator does not change the growth rate of the integral for z near the boundary. Thus, Theorem 3 implies that there is a constant $C > 0$ such that

$$|f(z) - f(0)| \leq C \| f \|_q (1 - \| z \|^2)^{-(q-1)}, \; z \in B.$$

This shows that $(1 - \| z \|^2)^{q-1} f(z)$ is bounded on B.

Conversely, if $(1 - \| z \|^2)^{q-1} |f(z)| \leq M$ for some constant $M > 0$, then

$$f(z) = c_q \int_B \frac{(1 - \| w \|^2)^{q-1}}{(1 - < z, w >)^{n+q}} f(w) d\nu(w)$$

by Theorem 1.

Differentiating under the integral sign, we obtain

$$\nabla f(z) =$$
$$c_q \int_B \frac{(n+q)(1 - < z, w >)^{n+q-1} (-\bar{w})(1 - \| w \|^2)^{q-1} f(w)}{(1 - < z, w >)^{2(n+q)}} d\nu(w),$$

$$\| \nabla f(z) \| \leq c_q (n+q) M \int_B \frac{1}{|1 - < z, w > |^{n+q+1}} d\nu(w).$$

By Theorem 3 , there exists a constant $C > 0$ such that

$$\| \nabla f(z) \| \leq \; CM(1 - \| z \|^2)^{-q}$$

for all $z \in B$. This clearly shows that f is in $\mathcal{B}_q$. □

3. Bounded Operator Q_q related with weighted Bloch spaces

Let N denote the set of natural numbers. A multi-index α is an ordered n-tuple $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ with $\alpha_j \in N, j = 1, 2, \cdots, n$. For a multi-index α and $z \in \mathbb{C}^n$, set

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$$

$$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$$

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2}\cdots z_n^{\alpha_n}$$

For any $q > 0$, let Q_q denote the operator defined by

$$Q_q f(z) = c_{q-1}\int_B \frac{f(w)}{(1-<z,w>)^{n+q}}d\nu(w), \quad z\in B.$$

Let $C_0(B)$ be the subspace of complex-valued continuous functions on B which vanish on the boundary, $C(\overline{B})$ the space of complex-valued continuous functions on the closed unit ball $\overline{B}$.

THEOREM 5. *For each $q > 0$, the opertor Q_q maps each function of the form $w^\alpha \overline{w}^\beta$ to a monomial.*

Proof. Since

$$\begin{aligned}
&< z, w >^m \\
&= (z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n)^m \\
&= \sum_{i_1+i_2+\cdots+i_n=m} \frac{m!}{i_1!i_2!\ldots i_n!}(z_1\overline{w}_1)^{i_1}(z_2\overline{w}_2)^{i_2}\ldots(z_n\overline{w}_n)^{i_n},
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{(1-<z,w>)^{n+q}} \\
&= 1 + (n+q) <z,w> + \frac{(n+q)(n+q+1)}{2!} <z,w>^2 \\
&\qquad + \frac{(n+q)(n+q+1)(n+q+2)}{3!} <z,w>^3 + \cdots \\
&= 1 + \sum_{m=1}^{\infty} \frac{(n+q+m-1)!}{m!(n+q-1)!} <z,w>^m \\
&= 1 + \sum_{m=1}^{\infty} \sum_{i_1+i_2+\cdots+i_n=m} \frac{(n+q+m-1)!}{m!(n+q-1)!}\frac{m!}{i_1!i_2!\ldots i_n!} \\
&\qquad (z_1\overline{w}_1)^{i_1}(z_2\overline{w}_2)^{i_2}\ldots(z_n\overline{w}_n)^{i_n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&Q_q(z^\alpha \overline{z}^\beta)\\
&= c_{q-1}\int_B \frac{w^\alpha \overline{w}^\beta}{(1-<z,w>)^{n+q}} d\nu(w)\\
&= c_{q-1}\int_B w^\alpha \overline{w}^\beta d\nu(w) + c_{q-1}\sum_{m=1}^{\infty}\sum_{i_1+i_2+\cdots+i_n=m}\\
&\qquad \frac{(n+q+m-1)!}{m!(n+q-1)!}\frac{m!}{i_1!i_2!\ldots i_n!} z^I \int_B w^\alpha \overline{w}^\beta \overline{w}^I d\nu(w),\\
&\qquad\qquad I=(i_1,i_2,\ldots,i_n)\\
&= c_J z^J \qquad \text{for some } J
\end{aligned}$$

by $[6, Prop. 1.4.8, Prop. 1.4.9]$. □

THEOREM 6. *For each $q > 0$, the operator Q_q maps $L^\infty(B)$ boundedly onto $\mathcal{B}_q$*

Proof. Let $f(z) = Q_q g(z)$, where $g \in L^\infty(B)$. Then

$$f(z) = c_{q-1}\int_B \frac{g(w)}{(1-<z,w>)^{n+q}} d\nu(w),$$

$$\nabla f(z) = (n+q)c_{q-1}\int_B \frac{g(w)(-\overline{w})}{(1-<z,w>)^{n+q+1}} d\nu(w),$$

$$\| \nabla f(z) \| \leq (n+q)c_{q-1} \| g \|_\infty \int_B \frac{d\nu(w)}{|1-<z,w>|^{n+q+1}} .$$

By Theorem 3,

$$\| \nabla f(z) \| \leq (n+q)c_{q-1} \| g \|_\infty (1-\| z \|^2)^{-q}.$$

Thus,

$$(1-\| z \|^2)^q \| \nabla f(z) \| \leq C \| g \|_\infty .$$

It is also clear that $|f(0)| \leq c_{q-1} \| g \|_\infty$. Thus,

$$\begin{aligned}
\| f \|_q &= |f(0)| + \sup\{(1-\| w \|^2)^q \| \nabla f(w) \| \,|w \in B\}\\
&\leq (C + c_{q-1}) \| g \|_\infty .
\end{aligned}$$

Hence, Q_q maps $L^\infty(B)$ boundedly into $\mathcal{B}_q$. □

THEOREM 7. *For each $q > 0$, the operator Q_q maps $C_0(B)$ onto $\mathcal{B}_q$*

Proof. Let $f \in \mathcal{B}_q \quad (q > 0)$. Then $(1- \| z \|^2)^{q-1}|f(z)|$ is bounded in B, by Theorem 4. By Theorem 1,

$$f(z) = c_{q-1} \int_B \frac{(1- \| w \|^2)^{q-1} f(w)}{(1- < z, w >)^{n+q}} d\nu(w) = Q_q h(z) \ ,$$

where $h(w) = (1- \| w \|^2)^{q-1} f(w)$ is in $C_0(B)$. Therefore, Q_q maps $C_0(B)$ onto $\mathcal{B}_q$. That Q_q maps $L^\infty(B)$ onto $\mathcal{B}_q$ is obvious. □

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DEPARTMENT OF MATHEMATICS
KONYANG UNIVERSITY
NONSAN 320-800, KOREA