JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **13**, No.1, June 2000

ALTERNATIVE PROOF OF EXISTENCE THEOREM FOR CERTAIN COMPETITION MODELS

Inkyung Ahn

Abstract. We give alternative proof of the existence theorem for certain elliptic systems describing competing interactions with nonlinear diffusion. The existence of positive solution depends on the sign of the principal eigenvalue of suitable operators of Schrödinger type. If the sign of such operators are both positive, then system has a positive solution. The main tool employed is the fixed point index of compact operator on positive cones.

1. Introduction

In [1], the author investigated the coexistence of positive solutions to the elliptic equations describing competing interactions between two species:

$$
\begin{cases} -\varphi(u,v)\Delta u = uf(u,v) & \\ -\psi(u,v)\Delta v = vg(u,v) & \text{in } \Omega \end{cases} \tag{1.1}
$$

under homogeneous Robin-type boundary condition. Ω is a bounded region in $\boldsymbol{R}^n$ with a smooth boundary and φ, ψ are strictly positive nondecreasing functions which denote the diffusion rates. Also u, v represent the densities of certain two species competing each other.

It was shown in [1] that the existence of positive solution of the system (1.1) under homogeneous Robin boundary condition depends on the sign of the principal eigenvalue of suitable operator of Schrödinger

This work is supported by KOSEF(K950701).
Received by the editors on May 1, 2000.
1991 *Mathematics Subject Classifications* : 35J60.
Key words and phrases: Competing interaction, Nonlinear diffusion, Fixed point index .

type. More precisely, positive solutions exist if the signs of the principal eigenvalues of those operators are positive. For the proof of the theorem, the method of decomposed operator was used. Also in [2], the coexistence of positive solutions is guaranteed even if the principal eigenvalue of the suitable operators are both negative. The main tool employed was the theorem concerning the fixed point index of compact operator on positive cones.

In this paper, we provide alternative proof of the existence theorem of the system (1.1) with competing interactions between two species under homogeneous Dirichlet boundary condition using the fixed point index theory.

2. Some lemmas and fixed point index

In this section, we state some known lemmas which are useful in the sequel. Throughout this paper, $\lambda_1(A)$ denotes the principal eigenvalue of an operator A on Ω under homogeneous Dirichlet boundary conditions.

The following lemma appears in [1].

LEMMA 2.1. *Let φ be C^1-function in u and C^α in x. Assume that φ is strictly positive, nondecreasing and concave down in u, and f is monotone nonincreasing C^1-function such that $f(x,0) > 0$ and concave down in u on the set (x,u) where $f(x,u) < 0$. If $\lambda_1(\varphi(x,0)\Delta + f(x,0)) > 0$, then the equation*

$$(2.1) \qquad \begin{cases} -\varphi(x,u)\Delta u = uf(x,u) & \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique positive solution in $C^{2,\alpha}(\bar{\Omega})$, $\alpha \in (0,1)$.

With the same assumptions in Lemma 2.1, there is a unique positive solution u_0 of the equation (2.1). Then we may linearize the equation (2.1) at $u = u_0 > 0$. Define the solution operator : $S \in C(\bar{\Omega})$ by

$S(u) = \bar{u}$, where $\bar{u}$ is the unique solution of

$$\begin{cases} -\varphi(x,\bar{u})\Delta\bar{u} + M\bar{u} = uf(x,u) + Mu \\ \bar{u} = 0 & \text{on } \partial\Omega \end{cases}$$

where $M > 0$ is sufficiently large. Note that $S(u_0) = u_0$. Define the operator S_L of linearization by $S_L(w) = v$, where v is the unique solution of

$$\begin{cases} -\varphi(x,u_0)\Delta v + Mv = wf(x,u_0) + wu_0 f_u(x,u_0) + Mw \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

LEMMA 2.2. *S is Fréchet differentiable at $u = u_0 \in C(\bar{\Omega})$ and $S'(u_0) = S_L$.*

Proof. Refer to Lemma 2.3 in [2] □

The next one is well-known result.

LEMMA 2.3. *Let $P > 0$ be constant. Suppose $a(x) \in C^1(\bar{\Omega})$ and $a(x) \geq \delta_0 > 0$, Consider for $\alpha \in (0,1)$,*

$$\begin{cases} -a(x)\Delta u + Pu = h(x), \quad h \in C^\alpha(\bar{\Omega}) \\ u = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Then (2.2) has a unique solution u in $C^{2,\alpha}(\bar{\Omega})$ and the solution operator T such that $u = Th$ is a compact operator in $\boldsymbol{X} = C(\bar{\Omega})$.

LEMMA 2.4. *The only solution to the linearized problem*

$$\begin{cases} -\varphi(x,u_0)\Delta w = w[f(x,u_0) + u_0 f_u(x,u_0)] \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

where u_0 is a unique solution to the equation

$$\begin{cases} -\varphi(x,u)\Delta u = uf(x,u) \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

is $w = 0$.

Proof. See Lemma 2.5 in [2]. □

Let $T : E \to E$ be a linear operator on a Banach space. Denote the spectral radius of T by $r(T)$.

LEMMA 2.5. *Assume that T is a compact positive linear operator on an ordered Banach space. Let $u > 0$ be a positive element. Then*
(i) *If $Tu > u$, then $r(T) > 1$.*
(ii) *If $Tu < u$, then $r(T) < 1$.*
(iii) *If $Tu = u$, then $r(T) = 1$.*

Proof. See Lemma 2.3 in [6].

Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a *wedge* if $\alpha W \subset W$ for all $\alpha \geq 0$. A wedge is said to be a *cone* if $W \cap (-W) = \{0\}$. For $y \in W$, define

$$W_y = \{x \in E \mid y + \gamma x \in W \text{ for some } \gamma > 0\}$$

$$S_y = \{x \in \overline{W}_y \mid -x \in \overline{W}_y\}.$$

Then $\overline{W}_y$ is a wedge containing W, y, $-y$, while S_y is a closed subspace of E containing y. Let T be a compact linear operator on E which satisfies $T(\overline{W}_y) \subset \overline{W}_y$. We say that T has a *property* α on $\overline{W}_y$ if there is a $t \in (0, 1)$ and a $w \in \overline{W}_y \backslash S_y$ such that $w - tTw \in S_y$.

Let $A : W \to W$ is a compact operator with fixed point $y \in W$ and A is Fréchet differentiable at y. Let $L = A'(y)$ be the Fréchet derivative of A at y. Then L maps $\overline{W}_y$ into itself.

For an open subset $U \subset W$, define $index(A, U, W) = deg_W(I - A, U, 0)$. To have deg_W well defined we require that W be a retract of E. By a result of Dugundji, every closed convex subset of real Banach space E is a retract of E. Since W is a wedge in E, W is a retract of E. We also have that S_y is a retract of E. Hence the above index is well defined. If y is an isolated fixed point of A, then the fixed point index of A at y in W is defined by $index_W(A, y) = index(A, y, W) = index(A, U(y), W)$, where $U(y)$ is a small open neighbourhood of y in W. We have the following theorem:

THEOREM 2.6. *Assume that $I - L$ is invertible on E.*

(i) *If L has property α on $\overline{W}_y$, then $index_W(A, y) = 0$.*

(ii) *If L does not have property α on $\overline{W}_y$, then $index_W(A, y) = (-1)^\sigma$, where σ is the sum of multiplicities of all the eigenvalues of L which are greater than 1.*

It many cases $I - L$ is not invertible on E, but is on $\overline{W}_y \backslash \{0\}$. For these cases, we have

(iii) *If $I-L : \overline{W}_y \to \overline{W}_y$ is not a surjective map, then $index_W(A, y) = 0$.*

(iv) *If L does not have* property *α on $\overline{W}_y$, then $index_W(A, y) = \pm 1$.*

Proof. This results in (i) (ii) are given by Dancer[4], and (iii)(iv) are supplements due to Li[6]. □

3. Steady-State Positive Solution of Competing Systems

In this section, we prove the existence theorem for the system :

$$\tag{3.1} \begin{cases} -\varphi(u, v)\Delta u = uf(u, v) & \\ -\psi(u, v)\Delta v = vg(u, v) & \text{in } \Omega \\ (u, v) = (0, 0) & \text{on } \partial\Omega. \end{cases}$$

For the system (3.1) with competing interactions, we assume the followings :

(H1) f, $g \in C^1(\mathrm{R}^+, \mathrm{R}^+)$ satisfy

$$f_u(u, v) < 0, \qquad f_v(u, v) < 0 \quad \text{for } u,\ v > 0$$

$$g_u(u, v) < 0, \qquad g_v(u, v) < 0 \quad \text{for } u,\ v > 0$$

(H2) There exist positive constants C_1, C_2 such that

$$f(u, 0) < 0 \quad \text{for } u > C_1$$

$$g(0, v) < 0 \quad \text{for } v > C_2.$$

(H3) $f(\cdot, v)$, $g(u, \cdot)$ are Lipschitz continuous and concave down for fixed u, $v \in \mathrm{R}^+$ respectively where $f(\cdot, v) < 0$, $g(u, \cdot) < 0$.

(H4) φ, ψ are strictly positive C^1-function in u, v, respectively, and nondecreasing, concave down in $u, v \in \mathrm{R}^+$.

(H1) shows that the system represents a competing interactions between two species and (H2) implies the logistic property of the growth rate of species. In (H3), the concavity of f and g does not effect the existence of positive solutions since the priori-bound of solutions u, v are C_1 and C_2, respectively.(See Theorem 4.1(i) below) Also note that the assumption (H4) covers the case φ, ψ are constants, which has been worked by many people.

By the Lemma 2.1, if $\lambda_1(\varphi(0,0)\Delta + f(0,0)) > 0$ in addition to (H1)-(H4), then there is a semi-trivial solution $(u_0, 0)$ to (3.1) where u_0 is the positive solution to

$$\begin{cases} \varphi(u)\Delta u + uf(u) = 0 & \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Similarly, if $\lambda_1(\psi(0,0)\Delta + g(0,0)) > 0$, then there is a semi-trivial solution $(0, v_0)$ to (3.1) where v_0 is the positive solution to

$$\begin{cases} \psi(v)\Delta v + vg(v) = 0 & \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

These solutions $(u_0, 0)$ and $(0, v_0)$ play an important role for the existence of positive solutions to the system (3.1).

Now we state the existence theorem of our system (3.1).

THEOREM 3.1. *Suppose that the assumptions (H1)-(H4) hold. Assume that* $\lambda_1(\varphi(0,0)\Delta + f(0,0)) > 0$ *and* $\lambda_1(\psi(0,0)\Delta + g(0,0)) > 0$.

(i) *If* (u, v) *is a strictly positive solution to (3.1), then*

$$0 < u(x) < u_0(x) < C_1, \quad 0 < v(x) < v_0(x) < C_2.$$

(ii) *If the principal eigenvalues of the operator* $\varphi(0, v_0)\Delta + f(0, v_0)I$ *and* $\psi(u_0, 0)\Delta + g(u_0, 0)I$ *are both positive, then the system (3.1) has a positive solution* (u, v).

Proof. For (i), refer to Lemma 5 in [1].

(ii) By continuity of the functions f, g, u, v on a compact set $\bar{\Omega}$, we can find $M > 0$ large enough so that

$$\max\{\max|f(u(x),v(x))|, \max|g(u(x),v(x))|\} < M.$$

For simplicity, we use the notations :

$$H(u,v) := (-\varphi(u,v)\Delta + M)^{-1}$$

$$R(u,v) := (-\psi(u,v)\Delta + M)^{-1}.$$

Define an operator :

$$A(u,v) := [H(\cdot,v)[uf(u,v) + Mu], R(u,\cdot)[vg(u,v) + Mv]].$$

Then A is the direct sum of positive compact operator. Note that system has a solution (u,v) if and only if (u,v) is a fixed point of A.

We introduce the following notations.

$$D := \{(u,v) \in C_0(\Omega) \oplus C_0(\Omega) \mid u \leq C_1 + 1,\ v \leq C_2 + 1\}$$

$$K := \{u \in C_0(\Omega) \mid 0 \leq u(x),\ x \in \overline{\Omega}\}$$

$$W := K \oplus K$$

$$P_\rho := \{(u,v) \in W \mid u \leq \rho,\ v \leq \rho\}, \quad \rho > 0$$

$$D' := (intD) \cap (K \oplus K).$$

Note that D' is open in W. To show that system has a strictly positive solution (u,v), we prove that A has a nontrivial fixed point in D'. So we need to calculate the fixed-point index for the trivial solution $(0,0)$ and semi-trivial solutions $(u_0,0)$ and $(0,v_0)$. We also require that the point be an isolated fixed point to use the fixed-point index for an operator at a point. Since we consider the operator A on the set D', if these fixed points are not isolated, then there must be a nontrivial fixed point in the interior of D' so that the system has a positive solution. Therefore we may assume that $(0,0)$, $(u_0,0)$ and $(0,v_0)$ are isolated fixed point of A.

We state the following lemma which one can find in [2].

LEMMA 3.2. *Assume* $\lambda_1(\varphi(0,0)\Delta + f(0,0)) > 0$ *and* $\lambda_1(\psi(0,0)\Delta + g(0,0)) > 0$. *Then*

(i) $index_W(A, D') = 1$

(ii) $index_W(A, (0,0)) = 0$

Now we claim the following lemma.

LEMMA 3.3. *Assume that* $\lambda_1(\varphi(0,0)\Delta + f(0,0)) > 0$ *and* $\lambda_1(\psi(0,0)\Delta + g(0,0)) > 0$. *If*

$$\begin{gathered}\lambda_1(\varphi(0,v_0)\Delta + f(0,v_0)I) > 0\\ \lambda_1(\psi(u_0,0)\Delta + g(u_0,0)I) > 0,\end{gathered}$$

then

$$index_W(A, (u_0,0)) = index_W(A, (0,v_0)) = 0.$$

Proof. It suffices to only calculate the index for the point $y = (u_0, 0)$ since the method for finding $index_W(A, (0, v_0))$ is the same as that used for finding $index_W(A, (u_0, 0))$ with the obvious notational changes.

For the point $y = (u_0, 0)$, observe that $\overline{W}_y = C_0(\Omega) \oplus K$ and by Lemma 2.2,

$$L := A'(u_0, 0)$$

$$= \begin{bmatrix} H(u_0,0)[f(u_0,0) + u_0 f_u(u_0,0) + M] & H(u_0,0)[u_0 f_v(u_0,0)] \\ 0 & R(u_0,0)[g(u_0,0) + M]. \end{bmatrix}$$

Let's show $index_W(A, (u_0, 0)) = 0$ using Theorem 2.6. We need consider two possibilities for the invertibility of $I - L$ on $C_0(\Omega) \oplus C_0(\Omega)$.

<u>Case 1</u> : Assume that $I - L$ is invertible on $C_0(\Omega) \oplus C_0(\Omega)$.

Claim: L has *property* α on $\overline{W}_y$.

Observe that

$$\begin{gathered} S_y = C_0(\Omega) \oplus \{0\} \\ \overline{W}_y \backslash S_y = C_0(\Omega) \oplus \{K \backslash \{0\}\}. \end{gathered}$$

Let $\mu := \lambda_1(\psi(u_0,0)\Delta + g(u_0,0)I)$. Then by the assumption, $\mu > 0$, so there exists a function $\phi_2 > 0$ such that

$$\begin{cases} \psi(u_0,0)\Delta\phi_2 + g(u_0,0)\phi_2 = \mu\phi_2 & \text{in } \Omega \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

This implies that

$$(-\psi(u_0,0)\Delta + M)^{-1}(g(u_0,0) + M)\phi_2 > \phi_2.$$

Let $T := R(u_0,0)(g(u_0,0) + M)I$. Then by Lemma 2.3, T is a compact positive operator and $T\phi_2 > \phi_2$. Thus $r(T) > 1$ by Lemma 2.5. By the Krein-Rutman Theorem, $r(T)$ is an eigenvalue of T with a corresponding positive eigenfunction $\phi_2 \in K\backslash\{0\}$. Set $t := \frac{1}{r(T)}$ and $\phi_1 \equiv 0$. Then $t \in (0,1)$ and $(\phi_1,\phi_2) \in \overline{W}_y \backslash S_y$ and

$$(I - tL)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{bmatrix} 0 \\ \phi_2 - tR(u_0,0)(g(u_0,0) + M)\phi_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \phi_2 - \frac{1}{r(T)}T\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence $(I - tL)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \in S_y$ and L has *property* α. Thus by Theorem 2.6 (i), we have that $index_W(A,(u_0,0)) = 0$.

Case 2 : Assume that $I - L$ is not invertible on $C_0(\Omega) \oplus C_0(\Omega)$.

Claim 1 : $I - L$ is invertible on $\overline{W}_y$.

Suppose there are functions $(\phi_1,\phi_2) \in \overline{W}_y$ such that

$$(I - L)\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Then we have

(3.2)

$$H(u_0,0)[f(u_0,0) + u_0 f_u(u_0,0) + M]\phi_1 + H(u_0,0)[u_0 f_v(u_0,0)]\phi_2 = \phi_1$$

$$R(u_0,0)[g(u_0,0)+M]\phi_2=\phi_2. \tag{3.3}$$

Note that the second equation (3.3) is equivalent to

$$\begin{cases} \psi(u_0,0)\Delta\phi_2+g(u_0,0)\phi_2=0 \\ \phi_2=0 \ \text{ on } \partial\Omega. \end{cases}$$

Since $\phi_2 \in K$, $\phi_2 \geq 0$ in Ω, we can consider ϕ_2 as eigenfunction of $\psi(u_0,0)\Delta+g(u_0,0)I$ and $\lambda_1(\psi(u_0,0)\Delta+g(u_0,0)I)=0$ which is a contradiction to our assumption. Thus $\phi_2\equiv 0$.

If we substitute this in (3.2), we have that

$$\begin{cases} \varphi(u_0,0)\Delta\phi_1+[f(u_0,0)+u_0f_u(u_0,0)]\phi_1=0 \\ \phi_1=0 \ \text{ on } \partial\Omega. \end{cases} \tag{3.4}$$

Observe that this equation (3.4) is the linearization of equation (2.3) at $u=u_0$. Thus by Lemma 2.4, $\phi_1\equiv 0$ is the only solution of (3.4). Therefore $I-L$ is invertible on $\overline{W}_y$.

Claim 2 : $I-L$ is not a surjective map on $\overline{W}_y$.

Since $I-L$ is not invertible on $C_0(\Omega)\oplus C_0(\Omega)$, there exist functions ϕ_1, $\phi_2\in C_0(\Omega)$, $(\phi_1,\phi_2)\not\equiv(0,0)$ such that

$$L\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}=\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}.$$

i.e.

$$H(u_0,0)[f(u_0,0)+u_0f_u(u_0,0)+M]\phi_1+H(u_0,0)[u_0f_v(u_0,0)]\phi_2=\phi_1$$

$$R(u_0,0)[g(u_0,0)+M]\phi_2=\phi_2.$$

These are equivalent to

$$\begin{cases} \varphi(u_0,0)\Delta\phi_1+[f(u_0,0)+u_0f_u(u_0,0)]\phi_1=-u_0f_v(u_0,0)\phi_2 \\ \psi(u_0,0)\Delta\phi_2+g(u_0,0)\phi_2=0 \\ (\phi_1,\phi_2)=(0,0) \quad \text{on } \partial\Omega. \end{cases}$$

In the second equation, first we assume $\phi_2 \equiv 0$. Then the first equation becomes

$$\begin{cases} \varphi(u_0,0)\Delta\phi_1 + [f(u_0,0) + u_0 f_u(u_0,0)]\phi_1 = 0 \\ \phi_1 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Note that this is the linearization of equation about $u = u_0$. Thus by Lemma 2.4, $\phi_1 \equiv 0$ is the only solution. Therefore $(\phi_1, \phi_2) \equiv (0,0)$, a contradiction. So $\phi_2 \not\equiv 0$. Let $\phi_0 \in C_0^\infty(\Omega)$, $\phi_0 > 0$ in Ω such that

$$\int_\Omega \phi_2\phi_0 \neq 0.$$

Let $\psi_0 = R(u_0,0)\phi_0$. Then $\psi_0 \in K$ since $R(u_0,0)$ is a positive operator. Let ψ_1 be an arbitrary function in $C_0(\Omega)$. Then $(\psi_1,\psi_0) \in \overline{W}_y$. We claim that (ψ_1,ψ_0) is not in the rage of $I - L$ on $\overline{W}_y$. Contrariwise assume that it is in the range. Then there exist functions $(r,s) \in C_0(\Omega) \oplus K$ such that

$$r - H(u_0,0)[f(u_0,0) + u_0 f_u(u_0,0) + M]r - H(u_0,0)u_0 f_v(u_0,0)s = \psi_1$$

$$s - R(u_0,0)[g(u_0,0) + M]s = \psi_0.$$

Then the second equation is equivalent to

$$-(\psi(u_0,0)\Delta + (g(u_0,0))I)s = (-\psi(u_0,0)\Delta + M)\psi_0 = \phi_0.$$

Multiply by ϕ_2 and integrate over Ω, then we have

$$-\int_\Omega \phi_2(\psi(u_0,0)\Delta + g(u_0,0)I)s = \int_\Omega \phi_2\phi_0 \neq 0.$$

Using integration by parts and the zero boundary condition on the left-hand side we obtain

$$-\int_\Omega s(\psi(u_0,0)\Delta + g(u_0,0)I)\phi_2 = 0,$$

which is a contradiction. Hence $I - L$ is not a surjective map on $\overline{W}_y$. Thus we may apply Theorem 2.6 (iii) to conclude $index_W(A,(u_0,0)) = 0$. $\square$

In the above two lemmas, we had the calculations of indices:

$$\begin{gathered} index_W(A, D') = 1 \\ index_W(A, (0,0)) = 0 \\ index_W(A, (u_0, 0) = 0 \\ index_W(A, (0, v_0)) = 0. \end{gathered}$$

So using the excision and solution properties for the index theory, we conclude that A has a nontrivial fixed point in D'. Therefore the system (3.1) has a strictly positive solution. □

References

1. I. Ahn, *Elliptic systems involving competing interactions with nonlinear diffusions*, Bull. Korean Math. Soc. 32, No. 1(1995), 123-132.
2. I. Ahn, *Elliptic systems involving competing interactions with nonlinear diffusions II*, Comm. Korean Math. Soc. 12, No 4(1997), 869-880.
3. H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. 18 (1976), 620-709.
4. E. N. Dancer, *On the indices of fixed points of mappings in cones and applications*, J. Math. Anal. Appl. 91(1983), 131-151.
5. D.E. Edmunds and W.P. Evans, *Spectral theory and differential operators*, Oxford Science Publication, 1987.
6. L. Li, *Coexistence theorems of steady-state for predator-prey interacting systems*, Trans. Amer. Math. Soc. 305 (1988), 143-166.

Department of Mathematics
Korea University
Chochiwon, Chungnam, 339-700, Korea
E-mail: ahn@gauss.korea.ac.kr