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ON STRICT EXTENSIONS OF STRONG δ-FRAMES

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ABSTRACT. In this paper, we introduce strict extensions on a δ-frame, and show that if L is a regular strong δ-frame, then $\mathcal{T}_X L$ is a regular δ-frame.

1. Introduction

The study of topological properties from a lattice-theoretic viewpoint was initiated by H. Wallman [14]. In particular, C. Ehresmann [5] and J. Bénabou [2] took the decisive step of regarding local lattices as generalized topological spaces in their own right. Such a local lattice is called a frame, a term introduced by C. H. Dowker and studied by D. Papert [4], J. R. Isbell [9], B. Banaschewski [1], P. T. Johnstone [10], Jorge Picado [11], and J. Wick Pelletier [15].

We note that continuous lattices and frames are characterized by certain distributive laws. We also note that a frame L is a complete lattice but in the theory of frames, we use only finite meets. Considering countable meets, we will get more properties of frames.

In this paper, a partially ordered set is also called a poset. If $\leq$ is a partial order on L, the smallest (largest, resp.) element of L, if it exists, is the element 0 (e, resp.) such that $0 \leq x$ ($x \leq e$, resp.) for each $x \in L$. Smallest (largest, resp.) elements are unique when they exist, by antisymmetry. We call 0 (e, resp.) the bottom (top, resp.) element of L. From now on, we denote a poset $(L, \leq)$ simply as L.

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DEFINITION 1.1. A map $f : S \to T$ between two posets S and T is called an *isotone* if for $a \le b$ in S, $f(a) \le f(b)$ in T.

DEFINITION 1.2. Let S and T be posets and let $f : S \to T$ and $g : T \to S$ be isotones. Then $(f,\ g)$ is said to be an *adjunction* or a *Galois connection* between S and T provided for any $x \in S$ and $y \in T$, $f(x) \le y$ iff $x \le g(y)$. In this case, f is said to be a *left adjoint* of g and g a *right adjoint* of f, and we write $f \dashv g$.

Let L be a poset. We say that L is a *complete lattice* if every subset A of L has the least upper bound and the greatest lower bound.

Every left adjoint preserves joins and every right adjoint preserves meets when they exist. Let S be a complete lattice and T a poset. Then $f : S \to T$ is a map which preserves joins iff f is an isotone and f has a right adjoint. Dually, if S is a complete lattice and T is a poset, then a map $f : S \to T$ has a left adjoint and f is an isotone iff f preserves meets.

DEFINITION 1.3. ([6]) A complete lattice L is called a *frame* (or *complete Heyting algebra*) if for any $a \in L$ and $S \subseteq L$,

$$a \wedge (\bigvee S) = \bigvee\{\ a \wedge s\ :\ s \in S\ \}.$$

DEFINITION 1.4. Let L be a frame. Then we say :

(1) For $A,\ B \subseteq L$, A *refines* B if for any $a \in A$, there is $b \in B$ with $a \le b$, which is denoted by $A \le B$.

(2) For $a,\ b \in L$, a is *well inside* b if there is $c \in L$ with $a \wedge c = 0$ and $c \vee b = e$. In case, we write $a \prec b$. Equivalently, $a \prec b$ iff $a^* \vee b = e$, where $a^* = \bigvee\{x \in L : a \wedge x = 0\}$, i.e., a^* is the pseudo complement of a.

(3) For $A \subseteq L$, A is a *cover* of L if $\bigvee A = e$. The set of all covers of L is denoted by $Cov(L)$ and

$$CCov(L) = \{\ A\ :\ \text{there is a countable cover } B \text{ with } B \leq A\}.$$

(4) L is *regular* if for any $a \in L,\ a = \bigvee\{x \in L : x \prec a\}$.

(5) For $x,\ y \in L$, we define $x \to y = \bigvee\{z \in L :\ x \wedge z \leq y\}$, so that $x \wedge z \leq y$ iff $z \leq x \to y$.

Definition 1.5. ([3]) A frame L is called a *δ-frame* if for any $a \in L$ and countable subset K of L,

$$a \vee (\bigwedge K) = \bigwedge\{a \vee k : k \in K\}.$$

Definition 1.6. ([3]) A frame L is called a *strong δ-frame* if for any countable family $(A_k)_{k\in\mathbb{N}}$ of subsets of L,

$$\bigwedge_{k\in\mathbb{N}} (\bigvee A_k) = \bigvee_{f \in \prod\limits_{k\in\mathbb{N}} A_k} (\bigwedge_{n\in\mathbb{N}} f(n)),$$

where $f = (f(n))_{n\in\mathbb{N}}$.

Every strong δ-frame is a δ-frame.

Definition 1.7. ([13]) Let L and M be frames (δ-frames, resp.). Then a map $f : L \longrightarrow M$ is called a *homomorphism* (*δ-homomorphism*, resp.) if f preserves a finite meets and arbitrary joins(countable meets and arbitrary joins, resp.).

Definition 1.8. A frame homomorphism $f : L \longrightarrow M$ is said to be :

(1) *open* if for $x,\ y \in L,\ f(x \to y) = f(x) \to f(y)$.
(2) *dense* if $f(x) = 0$ implies $x = 0$.

Definition 1.9. ([8]) Let L be a quasi ordered set and $F \subseteq L$. Then we say that F is a *filter* (δ-*filter*, resp.) on L if F satisfies the following :

(1) F does not contain 0.

(2) $F = \uparrow F = \{x \in L : a \leq x$ for some $a \in F\}$.

(3) For any finite (countable, resp.) subset K of F, there is $a \in F$ such that for all $x \in K,\ a \leq x$.

Proposition 1.10. *Let L be a strong δ-frame. Then $Cov(L)$ and $CCov(L)$ are δ-filters.*

Proof. It is trivial by Proposition 2.7 in [3]. □

For a set X of filters on a frame L, let $\mathcal{P}(X)$ denote the power set lattice of X and $L \times \mathcal{P}(X)$ the product frame of L and $\mathcal{P}(X)$. Then $\{(x, \sum) \in L \times \mathcal{P}(X)$: for any $F \in \sum,\ x \in F\}$ is a subframe of $L \times \mathcal{P}(X)$, which is denoted by $\mathcal{S}_X L$. And the restriction $s : \mathcal{S}_X L \to L$ of the first projection $Pr_1 : L \times \mathcal{P}(X) \to L$ is an onto, dense and open homomorphism.

For any $x \in L$, let $\sum_x = \{F \in X : x \in F\}$. Then the right adjoint s_* of s is given by $s_*(x) = (x, \sum_x)$ for any $x \in L$. Since s_* preserves meets, $s_*(L)$ is closed under finite meets ; and hence the subframe $\mathcal{T}_X L$ of $\mathcal{S}_X L$ generated by $s_*(L)$ is given by $\mathcal{T}_X L = \{ \bigvee\{(x, \sum_x) : x \in A\} : A \subseteq L\}$. Clearly the restriction $t : \mathcal{T}_X L \to L$ of s is an onto dense homomorphism ([7]).

2. Strict Extensions of a δ-Frame

Definition 2.1. Let L be a δ-frame and $M \subseteq L$. Then M is said to be a *sub-δ-frame* of L if M is closed under countable meets and arbitrary joins in L.

If M is a sub-δ-frame of a δ-frame L, then M is also a δ-frame.

Let L and M be δ-frames. For any subset $\{(x_\alpha,\ y_\alpha) : \alpha \in \Lambda\}$ of $L \times M$ and $\{(x_i,\ y_i) : i \in \mathbb{N}\} \in Count(L \times M)$, we define

$$\bigvee_{\alpha\in\Lambda} (x_\alpha,\ y_\alpha) = (\bigvee_{\alpha\in\Lambda} x_\alpha, \bigvee_{\alpha\in\Lambda} y_\alpha) \text{ and } \bigwedge_{i\in\mathbb{N}} (x_i,\ y_i) = (\bigwedge_{i\in\mathbb{N}} x_i, \bigwedge_{i\in N} y_i).$$

Then $L \times M$ is also a δ-frame.

Let L be a δ-frame and X a set of δ-filters in L. We will denote the power set lattice by $\mathcal{P}(X)$ in which the meet and the join are given by $\bigvee_{i\in I} A_i = \bigcup_{i\in I} A_i$ and $\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$ for any set I, respectively. Then $\mathcal{P}(X)$ is a δ-frame and hence $L \times \mathcal{P}(X)$ is a δ-frame. Let

$$\mathcal{S}_X L = \{(x,\ \sum) \in L \times \mathcal{P}(X) : \text{ for any } F \in \sum,\ x \in F\}.$$

Using this notation, the dense onto δ-homomorphism $s : \mathcal{S}_X L \longrightarrow L$ given by $(x,\ \sum) \longmapsto x$ is called the *simple extension* of L with respect to X([12]).

LEMMA 2.2. Let L be a δ-frame, X a set of δ-filters on L, and $s : \mathcal{S}_X L \to L$ a simple extension of L with respect to X. Then for any countable subset $\{x_i : i \in \mathbb{N}\}$ of L,

$$\sum_{\bigwedge_{i\in\mathbb{N}} x_i} = \bigcap_{i\in\mathbb{N}} \sum_{x_i}.$$

Proof. Since F is a δ-filter on L,

$$\begin{aligned} F \in \sum_{\bigwedge_{i\in\mathbb{N}} x_i} &\Longleftrightarrow \bigwedge_{i\in\mathbb{N}} x_i \in F \\ &\Longleftrightarrow x_i \in F \text{ for all } i \in \mathbb{N} \\ &\Longleftrightarrow F \in \sum_{x_i} \text{ for all } i \in \mathbb{N} \\ &\Longleftrightarrow F \in \bigcap_{i\in\mathbb{N}} \sum_{x_i}. \end{aligned}$$

□

THEOREM 2.3. *Let L be a strong δ-frame, X a set of δ-filters on L, and $s : \mathcal{S}_X L \to L$ a simple extension of L with respect to X. Let $\mathcal{B} = \{(x,\ \sum_x) : x \in L\}$. Then we have :*

(1) $\mathcal{B} = s_*(L)$.
(2) $\mathcal{B}$ *is closed under countable meets.*
(3) *The set* $\mathcal{T}_X L = \{\ \bigvee\{(x,\ \sum_x) : x \in A\} : A \subseteq L\}$ *is a sub-δ-frame of $S_X L$, which is generated by $\mathcal{B}$.*
(4) *If* $t : \mathcal{T}_X L \longrightarrow L$ *is defined by* $t(\bigvee\{(x,\ \sum_x) : x \in A\}) = \bigvee\{x : x \in A\}$ *for any* $\bigvee\{(x,\ \sum_x) : x \in A\} \in \mathcal{T}_X L$, *then t is a dense onto δ-homomorphism.*

Proof. (1) By the definition of $s_* : L \to \mathcal{S}_X L$, it is trivial.

(2) For any countable subset $\{(x_i,\ \sum_{x_i}) : i \in \mathbb{N}\}$ of $\mathcal{B}$, by Lemma 2.2,

$$\bigwedge_{i\in\mathbb{N}} (x_i,\ \textstyle\sum_{x_i}) = (\bigwedge_{i\in\mathbb{N}} x_i,\ \bigcap_{i\in\mathbb{N}} \textstyle\sum_{x_i}) = (\bigwedge_{i\in\mathbb{N}} x_i,\ \textstyle\sum_{\bigwedge_{i\in\mathbb{N}} x_i}) \in \mathcal{B}.$$

(3) Since $\mathcal{T}_X L$ is a subframe of $\mathcal{S}_X L$, it is enough to show that for any countable subset $\{(y_i,\ \Lambda_i) : i \in \mathbb{N}\}$ of $\mathcal{T}_X L$, $\bigwedge_{i\in\mathbb{N}} (y_i,\ \Lambda_i) \in \mathcal{T}_X L$. Take any countable subset $\{(y_i,\ \Lambda_i) : i \in \mathbb{N}\}$ of $\mathcal{T}_X L$. Then there is a subset A_i of L such that $(y_i,\ \Lambda_i) = \bigvee_{x\in A_i} (x,\ \sum_x)$ for each $i \in \mathbb{N}$. Thus we have

$$\begin{aligned}
\bigwedge_{i\in\mathbb{N}} (y_i,\ \Lambda_i) &= \bigwedge_{i\in\mathbb{N}} (\bigvee_{x\in A_i} (x,\ \textstyle\sum_x)) \\
&= \bigwedge_{i\in\mathbb{N}} (\bigvee_{x\in A_i} x,\ \bigcup_{x\in A_i} \textstyle\sum_x) \\
&= \bigwedge_{i\in\mathbb{N}} (\bigvee A_i,\ \textstyle\sum_{A_i}) \\
&= (\bigwedge_{i\in\mathbb{N}} (\bigvee A_i),\ \bigcap_{i\in\mathbb{N}} \textstyle\sum_{A_i}).
\end{aligned}$$

Since L is a strong δ-frame,

$$\bigwedge_{i\in\mathbb{N}}(\bigvee A_i) = \bigvee_{f\in\prod_{i\in\mathbb{N}} A_i}(\bigwedge_{i\in\mathbb{N}} f(i)).$$

On the other hand,

$$\begin{aligned}\bigcap_{i\in\mathbb{N}}\Sigma_{A_i} &= \bigcap_{i\in\mathbb{N}}(\bigcup_{x\in A_i}\Sigma_x)\\ &= \bigcup_{f\in\prod_{i\in\mathbb{N}} A_i}(\bigcap_{i\in\mathbb{N}}\Sigma_{f(i)})\\ &= \bigcup_{f\in\prod_{i\in\mathbb{N}} A_i}(\Sigma_{\bigwedge_{i\in\mathbb{N}} f(i)}).\end{aligned}$$

Collecting these, we have

$$\begin{aligned}\bigwedge_{i\in\mathbb{N}}(y_i,\ \Lambda_i) &= (\bigvee_{f\in\prod_{i\in\mathbb{N}} A_i}(\bigwedge_{i\in\mathbb{N}} f(i)),\ \bigcup_{f\in\prod_{i\in\mathbb{N}} A_i}(\Sigma_{\bigwedge_{i\in\mathbb{N}} f(i)}))\\ &= \bigvee_{f\in\prod_{i\in\mathbb{N}} A_i}(\bigwedge_{i\in\mathbb{N}} f(i),\ \Sigma_{\bigwedge_{i\in\mathbb{N}} f(i)}) \in \mathcal{T}_X L.\end{aligned}$$

Thus $\mathcal{T}_X L$ is a sub-δ-frame of $\mathcal{S}_X L$.

(4) Since s is a dense onto δ-homomorphism and $\mathcal{T}_X L$ is a sub-δ-frame of $\mathcal{S}_X L$, it is immediate. □

The following definition is derived from Theorem 2.3.

DEFINITION 2.4. The dense onto δ-homomorphism $t : \mathcal{T}_X L \longrightarrow L$ is called a *strict extension* of L with respect to X.

It is easily shown that a strict extension t in the above definition has a right adjoint t_*, where $t_* : L \to \mathcal{T}_X L$ is given by $t_*(x) = (x,\ \Sigma_x)$.

Remark 2.5. Let L be a strong δ-frame, X a set of δ-filters on L and $s : \mathcal{S}_X L \to L$ the simple extension of L with respect to X. Then for the strict extension $t : \mathcal{T}_X L \longrightarrow L$ with $t \dashv t_*$ and $x \in L$, $(x,\ \sum_x)^* = (x^*,\ \sum_{x^*})$.

We recall that for $(a,\ A),\ (b,\ B) \in L \times \mathcal{P}(X)$, $(a,\ A) \prec (b,\ B)$ if there is $(c,\ C) \in L \times \mathcal{P}(X)$ with $(a,\ A) \wedge (c,\ C) = (0,\ \emptyset)$ and $(c,\ C) \vee (b,\ B) = (e,\ X)$. Equivalently, $(a,\ A) \prec (b,\ B)$ iff $(a,\ A)^* \vee (b,\ B) = (e,\ X)$, where $(a,\ A)^*$ is the pseudo complement of $(a,\ A)$. In fact,

$$(a,\ A)^* = (a^*,\ A^c).$$

Theorem 2.6. *Let L be a strong δ-frame and $X = \{F : F$ is a δ-filter on L and $a \prec b$ implies $a^* \in F$ or $b \in F\}$. Then we have :*

(1) *$a \prec b$ implies $(a,\ \sum_a) \prec (b,\ \sum_b)$ in $\mathcal{T}_X L$.*

(2) *If L is a regular, then $\mathcal{T}_X L$ is a regular δ-frame.*

Proof. (1) Let $a \prec b$. Then by Remark 2.5, $(a,\ \sum_a)^* \vee (b,\ \sum_b) = (a^* \vee b,\ \sum_{a^*} \bigcup \sum_b)$. To show that $X = \sum_{a^*} \bigcup \sum_b$, take any $F \in X$. Then $a^* \in F$ or $b \in F$; hence $F \in \sum_{a^*}$ or $F \in \sum_b$. Hence $F \in \sum_{a^*} \bigcup \sum_b$. Thus $X = \sum_{a^*} \bigcup \sum_b$. Therefore, $(a,\ \sum_a) \prec (b,\ \sum_b)$.

(2) Take any $(a,\ \sum_a) \in \mathcal{T}_X L = [\mathcal{B}]$. Since L is regular, $a = \bigvee\{x \in L : x \prec a\}$. Then $\sum_a = \bigcup\{\sum_x : x \prec a\}$. Hence by (1),

$$\begin{aligned}(a,\ \textstyle\sum_a) &= (\bigvee\{x \in L : x \prec a\},\ \bigcup\{\textstyle\sum_x : x \prec a\})\\ &= \bigvee\{(x,\ \textstyle\sum_x) : x \prec a\}\\ &\leq \bigvee\{(x,\ \textstyle\sum_x) : (x,\ \textstyle\sum_x) \prec (a,\ \textstyle\sum_a)\}\\ &\leq (a,\ \textstyle\sum_a).\end{aligned}$$

Thus $\mathcal{T}_X L$ is regular. □

THEOREM 2.7. *Let L be a strong δ-frame and $X = \{F : F$ is a δ-filter on L and $a \prec b$ implies $a^* \in F$ or $b \in F\}$. Then $a \prec b$ in L iff $\sum_a \prec \sum_b$ in $\mathcal{T}_X L$.*

Proof. Let $a \prec b$. Since $\sum_a \bigvee \sum_b \subset \sum_{a^* \vee b}$, we will show that $\sum_{a^* \vee b} \subset \sum_{a^*} \bigvee \sum_b$. If $F \in \sum_{a^* \vee b} \subset X$, then $a^* \in F$ or $b \in F$. Thus $F \in \sum_{a^*} \bigvee \sum_b$. Hence $X = \sum_e = \sum_{a^* \vee b} = \sum_{a^*} \bigvee \sum_b = \sum_a^* \bigvee \sum_b$; and hence $\sum_a \prec \sum_b$. The other hands, $\sum_a \prec \sum_b$ implies $X = \sum_a^* \bigvee \sum_b = \sum_{a^*} \bigvee \sum_b \subset \sum_{a^* \vee b}$ and hence $\sum_{a^* \vee b} = X$. Then $a^* \vee b = e$. Thus $a \prec b$. □

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