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A NOTE ON RECURSIVE SETS FOR MAPS OF THE CIRCLE

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Abstract. For a continuous map f of the circle to itself, we show that if $P(f)$ is closed, then $\Gamma(f)$ is closed , and $\Omega(f) = \Omega(f^n)$ for all $n > 0$.

1. Introduction

Let I be the unit interval, S^1 the circle. Throughout this paper f denote a continuous map from the circle S^1 into itself.

For any positive integer n, we define f^n inductively by $f^1 = f$ and $f^{n+1} = f \circ f^n$, and let f^0 denote the identity map of S^1. Let $P(f), R(f), \Gamma(f), \Lambda(f)$ and $\Omega(f)$ denote the collection of the periodic points, recurrent points, γ-limit points, ω-limit points and nonwandering points of f, respectively. And for $A \subset S^1$, the set of all accumulaiton points of A denoted by A'.For any continuous map g from the interval I into itself, E.M.Coven and Z.Nitecki [6] proved that $\Omega(g) = \Omega(g^n)$ for each odd integer $n > 0$, moreover, $\bigcap_{n=1}^{\infty} \Omega(g^n) = \bigcap_{n=1}^{\infty} \Omega(g^{2^n})$, and that $\Omega(g^2)$ can be a proper subset of $\Omega(g)$. Furthermore, $\Omega(g) \setminus \bigcap_{n=1}^{\infty} \Omega(g^n)$ cannot be empty. And J.C.Xiong [7] proved that the set $\Omega(g) \setminus \bigcap_{n=1}^{\infty} \Omega(g^n)$ is small, that is $\Omega'(g) \subset \bigcap_{n=1}^{\infty} \Omega(g^n)$.

On the other hand, J.S. Bae and S. K. Yang [1] proved that $R(f) = R(f^n)$ for all integer $n > 0$ and S.H. Cho [5] proved that $\Lambda(f) = \Lambda(f^n)$

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for all integer $n > 0$. Also L.Block, E. M. Coven, I. Mulvey and Z.Nitecki [4] proved that $\Omega(f) = \Omega(f^n)$ for each odd integer $n > 0$.

In this paper, we have

THEOREM 2. *If $P(f)$ is closed, then $\Omega(f) = \Omega(f^n)$ for all $n > 0$.*

2. Preliminaries and Definitions

A point $x \in S^1$ is called a *periodic point* of f if for some positive integer n, $f^n(x) = x$. The period of x is the least such integer n. We denote the set of periodic points of f by $P(f)$.

A point $x \in S^1$ is called a *recurrent point* of f if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to x$. We denote the set of recurrent points of f by $R(f)$.

A point $x \in S^1$ is called a *nonwandering point* of f if for every neighborhood U of x, there exists a positive integer m such that $f^m(U) \cap U \neq \emptyset$. We denote the set of nonwandering points of f by $\Omega(f)$.

A point $y \in S^1$ is called an *ω-limit point* of x if there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x) \to y$. We denote the set of ω-limit points of x by $\omega(x, f)$. and define

$$\Lambda(f) = \bigcup_{x \in S^1} \omega(x, f).$$

A point $y \in S^1$ is called an *α-limit point* of x if there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{y_i\}$ of points such that $f^{n_i}(y_i) = x$ and $y_i \to y$. The symbol $\alpha(x, f)$ denotes the set of α-limit points of x.

A point $y \in S^1$ is called a *γ-limit point* of x if $y \in \omega(x, f) \cap \alpha(x, f)$. The symbol $\gamma(x, f)$ denotes the set of γ-limit points of x and define

$$\Gamma(f) = \bigcup_{x \in S^1} \gamma(x, f).$$

Let $a, b \in S^1$ with $a \neq b$. We denote the open (*closed, half-open*) arc from a counterclockwise to b by (a, b) (resp., $[a, b], (a, b]$). In particular, for $a, b, c \in S^1$, $a < b < c$ means that b lies in the open arc (a, c).

Now, we will use the symbols $\omega_+(x, f)$ (*resp.* $\omega_-(x, f)$) to denote the set of all points $y \in S^1$ such that there exists a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ such that $f^{n_i}(x, f) \to y$ and $y < \cdots < f^{n_i}(x, f) < \cdots < f^{n_2}(x) < f^{n_1}(x)$ (*resp.* $f^{n_1}(x) < f^{n_2}(x) < \cdots < f^{n_i}(x) < \cdots < y$). It is clear that if $x \notin P(f)$, then $\omega(x, f) = \omega_+(x, f) \cup \omega_-(x, f)$. Define $\Lambda_+(f) = \bigcup_{x \in S^1} \omega_+(x, f)$ and $\Lambda_-(f) = \bigcup_{x \in S^1} \omega_-(x, f)$.

Also, we will use the symbols $\alpha_+(x, f)$ (*resp.* $\alpha_-(x, f)$) to denote the set of all points $y \in S^1$ such that there exist a sequence $\{n_i\}$ of positive integers with $n_i \to \infty$ and a sequence $\{x_i\}$ of points such that $x_i \to y$, $f^{n_i}(x_i) = x$ for every $i > 0$, and $y < \cdots < x_i < \cdots < x_2 < x_1$ (*resp.* $x_1 < x_2 < \cdots < x_i < \cdots < y$). It is clear that if $x \notin P(f)$, then $\alpha(x, f) = \alpha_+(x, f) \cup \alpha_-(x, f)$.

Define

$$\gamma_+(x, f) = \omega_+(x, f) \cap \alpha_+(x, f)$$

$$\gamma_-(x, f) = \omega_-(x, f) \cap \alpha_-(x, f)$$

$$\Gamma_+(f) = \bigcup_{x \in S^1} \gamma_+(x, f)$$

$$\Gamma_-(f) = \bigcup_{x \in S^1} \gamma_-(x, f)$$

The following lemmas are found in [2].

LEMMA 1. $P(f) \subset R(f) \subset \Gamma(f) \subset \overline{R(f)} \subset \Lambda(f) \subset \Omega(f)$.

LEMMA 2. $\Gamma(f) \subset R(f) \cup \overline{P(f)}$.

Let J, K be two closed arcs in S^1. We say that J f-covers K if there is a closed arc $L \subset J$ such that $f(L) = K$. In [3], J.S.Bae et al obtained some result for the maps of the circle by using the concepts of f-covering as follows;

LEMMA 3. *Let $J = [a, b]$ be an arc for some $a, b \in S^1$ with $a \neq b$, and let $J \cap P(f) = \emptyset$.*

(a) Suppose there exists $x \in J$ such that $f(x) \in J$ and $x < f(x)$. Then

(1) if $y \in J, x < y$ and $f(y) \notin [y, b]$, then $[x, y]$ f-covers $[f(x), b]$,

(2) if $y \in J, x > y$ and $f(y) \notin [y, b]$, then $[y, x]$ f-covers $[f(x), b]$.

(b) Suppose there exists $x \in J$ such that $f(x) \in J$ and $x > f(x)$. Then

(1) if $y \in J, x < y$ and $f(y) \notin [a, y]$, then $[x, y]$ f-covers $[a, f(x)]$,

(2) if $y \in J, y < x$ and $f(y) \notin [a, y]$, then $[y, x]$ f-covers $[a, f(x)]$.

LEMMA 4. *$x \in \Omega(f)$ if and only if $x \in \alpha(x, f)$.*

3. Main results

PROPOSITION 1. *If $P(f)$ is a closed set, then $\Gamma(f) = \Gamma(f^n)$ for all $n \geq 1$.*

Proof. By definition, we know $\Gamma(f^n) \subset \Gamma(f)$ for all $n \geq 1$. Suppose that $P(f)$ is closed. Then we have $\Gamma(f) = R(f)$ by Lemma 1 and Lemma 2. Since $R(f) = R(f^n)$, we have

$$\Gamma(f) = R(f) = R(f^n) = \Gamma(f^n).$$

Therefore $\Gamma(f) = \Gamma(f^n)$ for all $n \geq 1$. □

The following lemma follows from [1].

LEMMA 5. *If $P(f) \neq \emptyset$, then $\overline{P(f)} = \overline{R(f)}$.*

THEOREM 1. *If $P(f)$ is a closed set, then $\Gamma(f)$ is closed.*

Proof. If $P(f) \neq \emptyset$, then, by Lemma1,2 and 5,we have

$$\begin{aligned}\Gamma(f) \subset R(f) \cup \overline{P(f)} &= R(f) \cup \overline{R(f)} \\ &= \overline{R(f)} = \overline{P(f)} = P(f) \subset \Gamma(f).\end{aligned}$$

Thus $\Gamma(f)$ is closed .

Assume that $P(f) = \emptyset$. It is well known that $R(f) \neq \emptyset$. So $\Gamma(f) \neq \emptyset$, moreover $\overline{\Gamma(f)} \neq \emptyset$. Suppose that $x \in \overline{\Gamma(f)} \setminus \Gamma(f)$, that is, $x \in \Gamma'(f)$ and $x \notin \Gamma(f)$. Since $x \notin R(f)$, there exists an open arc (a, b) in S^1 containing x such that $f^n(x) \notin (a, b)$ for all $n \geq 1$. We may assume that there exists a sequence $\{x_i\} \in \Gamma(f)$ with $a < x_1 < x_2 < \cdots < x_i < \cdots < x < b$ such that $x_i \to x$. For each i, there exist n_i and m_i with $n_i < m_i$ such that either

$$x_{i-1} < f^{n_i}(x_i) < f^{m_i}(x_i) < x_i$$

or

$$x_i < f^{m_i}(x_i) < f^{n_i}(x_i) < x_{i+1}$$

Therefore, for each $i = 1,\ 2,\ \cdots$, there exist $y_i,\ z_i \in (x_{i-1}, x_{i+1})$ and n_i, m_i with $n_i < m_i$ such that

$$x_{i-1} < f^{n_i}(y_i) < y_i < x_i < x$$

and

$$x_i < z_i < f^{m_i}(z_i) < x_{i+1} < x.$$

By Lemma 3, $[y_i, x]$ f^{n_i} -covers $[a, f^{n_i}(y_i)]$ and $[z_i, x]$ f^{m_i} -covers $[f^{m_i}(z_i), b]$. Therefore, for each $i = 1, 2, \cdots$, we have

(i) $$[x_{i-1}, x] \ f^{n_i} \text{ -covers } \ [x_1, x_{i-1}]$$

and

(ii) $$[x_{i-1}, x] \ \ f^{m_i}\text{ -covers } \ [x_{i+1}, x].$$

Since $\bigcap_{i=2}^{\infty}([x_{i-1}, x]) = \{x\}$, there exists $z \in [x_1, x)$ such that $f^{l_i}(z) \in [x_{i+1}, x]$ and $f^{l_i}(z) \to x$ as $i \to \infty$ with $l_i = \sum_{k=1}^{i} n_k$. Hence $x \in \omega(z)$. Now, take N with $x_{N-1} > z$. By (i), for all $i \geq N$, there exists $u_i \in [x_{i-1}, x]$ such that $f^{n_i}(u_i) = z$. Since $x_i \to x$, we have $u_i \to x$, and hence $x \in \alpha(z, f)$. Therefore $x \in \Gamma(f)$, a contradiction. The proof of Theorem is complete. □

COROLLARY 1. *If $P(f)$ is a closed set, then $\Gamma'(f) \subset \bigcap_{n=1}^{\infty} \Gamma(f^n)$ for all $n \geq 1$.*

PROPOSITION 2. [3]. *If $P(f) = \emptyset$, then $R(f)$ is closed.*

PROPOSITION 3. [5]. *$\Lambda(f) = \Lambda(f^n)$ for all $n \geq 1$.*

PROPOSITION 4. [4]. *If $P(f)$ is closed and non-empty, then*

$$\Omega(f) = P(f).$$

THEOREM 2. *If $P(f)$ is closed, then $\Omega(f) = \Omega(f^n)$ for all $n \geq 1$.*

Proof. Obviously, $\Omega(f^n) \subset \Omega(f)$ by definition. Now we show that $\Omega(f) \subset \Omega(f^n)$. If $P(f)$ is closed and non-empty, then $\Omega(f) = \Lambda(f) = \Lambda(f^n) \subset \Omega(f^n)$ by Lemma 1, Proposition 3 and 4. Thus $\Omega(f) = \Omega(f^n)$ for all $n \geq 1$. Assume that $P(f) = \emptyset$. Let $x \in \Omega(f) \setminus \Omega(f^n)$. Then

$$\begin{aligned} \Omega(f) \setminus \Omega(f^n) &\subset \Omega(f) \setminus \Lambda(f^n) \text{ by Lemma 1} \\ &\subset \Omega(f) \setminus \Lambda(f) \text{ by Proposition 3} \\ &\subset \Omega(f) \setminus \overline{R(f)} \text{ by Lemma 1} \end{aligned}$$

Hence $x \in \Omega(f) \setminus \overline{R(f)}$. Let C be a connected component of $S^1 \setminus \overline{R(f)}$ containing x. Then there exist $a, b \in S^1$ with $a \neq b$ such that

$C = (a, b)$. Since $x \in \Omega(f)$, $x \in \alpha(x, f)$ by Lemma 4. Without loss of generality, we may assume that $x \in \alpha_+(x, f)$. Then there exist $z \in S^1$ with $x < z < b$ and positive integer m such that $f^m(z) = x$. Then we know that $a < x = f^m(z) < z < b$, and that $a \in R(f)$ by Proposition 2. Since $R(f)$ is invariant under f, $a \in \omega_-(a, f)$. Then we have know that $[a, z]$ f^m- covers $[f^m(a), x]$, and $f^m(a) < a < x$. Since $a \in \omega_-(a, f)$, there exist positive integers l, n such that $f^m(a) < f^l(a) < f^{n+l}(a) < a < x$. Especially, $[a, z]$ f^m- covers $[f^l(a), a]$. Since $f^n(a) \notin (a, b)$ and $f^n(a) \in [b, a]$, $[f^l(a), a]$ f^n-covers $[f^{l+n}(a), b]$, and hence $[f^l(a), a]$ f^n-covers $[a, z]$. Therefore we have $[a, z]$ f^{m+n} -covers itself, and hence f has a periodic point in $[a, z]$. This is a contradiction. The proof is completed. □

It is well known that $\Omega(f)$ is closed. So we have the following corollary.

COROLLARY 2. *If $P(f)$ is closed , then $\Omega^{'}(f) \subset \bigcap_{i=1}^{\infty} \Omega(f^i)$.*

REFERENCES

[1] J.S. Bae and S.K.Yang, *$\overline{P} = \overline{R}$ for maps of the circle*, Bull. Korean Math. Soc. **24** (1998).

[2] J.S.Bae, S.H.Cho and S.K.Yang, *Relations between nonwandering points and turning points on the circle*, Bull. Korean Math. Soc. **32** (1995), 57-66.

[3] J.S.Bae, S.H. Cho, K.J.Min and S.K.Yang, *Nonwandering points of a map on the circle*, J.Korean Math. Soc **33** (1990), 1115-1121.

[4] L.Block, E.M. Coven, I.Mulvey and Z.Nitecki, *Homoclinic and nonwandering points for maps of the circle*, Erg. Th and Dyn. Sys **3** (1983), 521-532.

[5] S.H. Cho, *Nonwandering sets of the powers on the circle*, J. chung Cheong Math. Soc **9** (1996), 107-113.

[6] E.M.Coven and Z.Nitecki, *Nonwandering set of the powers of the interval*, Erg. Th and Dyn. Sys **1** (1981), 9-13.

[7] J. Xiong, *A note on nonwandering set of the powers of the interval self-maps, preprint.*

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