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PROPERTIES OF D–WEAKLY CONTINUOUS FUNCTIONS

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Abstract. We introduce the notion of d-weakly continuous function, and investigate some of their properties.

1. Introduction

Continuity is of fundamental importance in topology. In this paper, we introduce the notion of d-weakly continuous function which is more general than that of the continuous function, and study some properties of d-weakly continuous functions.

2. D-weakly continuous

The notation $\mathrm{cl}(A)$ denote the closure of the subset A in a topological space. Throughout this paper, the symbols X, Y and Z denote topological spaces where no separation axioms are assumed unless explicitly stated.

Definition 2.1. Let X be a space and let A be a subset of X. The w-closure $\mathrm{cl}_w(A)$ of A is defined by

$$\mathrm{cl}_w(A) = \{x \in X \mid A \cap \mathrm{cl}(U) \neq \phi \text{ for all neighborhoods } U \text{ of } x \}.$$

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DEFINITION 2.2. A function $f : X \to Y$ is d-weakly continuous at a point x in X if for each neighborhood of $f(x)$, there is a neighborhood U of x such that $f(U) \subset \mathrm{cl}_w(A)$. A function $f : X \to Y$ is d-weakly continuous if it is d-weakly continuous at each point of X.

A continuous function implies d-weakly continuous function, but as the following example shows, the converse does not necessarily hold.

EXAMPLE 2.3. [5] Let X be the set of all real numbers with the cofinite topology and let $Y = \{0, 1, 2\}$ with the topology $\{\phi, \{0\}, \{2\}, \{0, 2\}, Y\}$. Define a function $f : X \to Y$ by $f(x) = 0$ if x is rational and $f(x) = 1$ if x is irrational. Then f is d-weakly continuous but not continuous.

Whenever the range of a function is a regular space, the following theorem holds.

THEOREM 2.4. *If a function $f : X \to Y$ is d-weakly continuous and Y is regular, then f is continuous.*

Proof. Let $x \in X$ and let G be any neighborhood of $f(x)$. Then since Y is regular, there is a neighborhood V of $f(x)$ such that $\mathrm{cl}_w(V) \subset G$. Since f is d-weakly continuous at x, there is a neighborhood U of x such that $f(U) \subset \mathrm{cl}_w(V)$. Therefore we have $f(U) \subset G$. Hence f is continuous. □

D-weakly continuous function can be characterized by the inverse image of open sets under f as follows.

THEOREM 2.5. *A function $f : X \to Y$ is d-weakly continuous if and only if for any open set V of Y, $f^{-1}(V) \subset f^{-1}(cl_w(V))^o$.*

Proof. Suppose that f is d-weakly continuous. Let V be any open set in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$ and so V is a neighborhood of $f(x)$. Since f is d-weakly continuous at x, there is a neighborhood U of x such that $f(U) \subset \mathrm{cl}_w(V)$. Since $U \subset f^{-1}(f(U)) \subset f^{-1}(\mathrm{cl}_w(V))$ and U is open, we have $U \subset f^{-1}(\mathrm{cl}_w(V))^o$. Thus $x \in f^{-1}(\mathrm{cl}_w(V))^o$ and so we obtain $f^{-1}(V) \subset f^{-1}(\mathrm{cl}_w(V))^o$.

Conversely, suppose that for any open set V of Y, $f^{-1}(V) \subset f^{-1}(\mathrm{cl}_w(V))^o$. Let $x \in X$ and let G be any neighborhood of $f(x)$. Then since G is open in Y, by hypothesis we have

$$f^{-1}(G) \subset f^{-1}(\mathrm{cl}_w(G))^o.$$

Let $U = f^{-1}(\mathrm{cl}_w(G))^o$. Since $x \in f^{-1}(G), U$ is a neighborhood of x. Since $f(U) = f(f^{-1}(\mathrm{cl}_w(G))^o) \subset f(f^{-1}(\mathrm{cl}_w(G))) \subset \mathrm{cl}_w(G), f$ is d-weakly continuous at x. Hence the proof is complete. □

THEOREM 2.6. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is d-weakly continuous, then $g \circ f : X \to Z$ is d-weakly continuous.*

Proof. Let $x \in X$ and let G be any neighborhood of $g(f(x))$. Then since g is d-weakly continuous at $f(x)$, there is a neighborhood V of $f(x)$ such that $g(V) \subset \mathrm{cl}_w(G)$. Also, since f is continuous at x, there exists a neighborhood U of x such that $f(U) \subset V$. Since $g(f(U)) \subset g(V) \subset \mathrm{cl}_w(G)$, $g \circ f$ is d-weakly continuous at x. This completes the proof. □

We know the restriction of a continuous function on a subset is continuous. Also, this fact holds for d-weakly continuous function.

THEOREM 2.7. *If $f : X \to Y$ is d-weakly continuous and A is a subspace of X, then $f|_A : A \to Y$ is d-weakly continuous.*

Proof. Let $x \in A$ and let G be any neighborhood of $f|_A(x) = f(x)$. Then since f is d-weakly continuous at x, there is a neighborhood V of x in X such that $f(V) \subset \mathrm{cl}_w(G)$. Let $U = V \cap A$. Then U is a neighborhood of x in A. Since $f|_A(U) = f(U) \subset f(V) \subset \mathrm{cl}_w(G), f|_A$

is d-weakly continuous at x. Hence the proof of the theorem is complete. □

We introduce the theorem which will use to prove the following theorem.

THEOREM 2.8. [4] *Let Y be an open or dense subspace of X and let A be a subset of Y. Then w-closure $cl_{w_Y}(A)$ of A in Y equals $cl_w(A) \cap Y$.*

THEOREM 2.9. *If $f : X \to Y$ is d-weakly continuous and Z is a space having Y as an open subspace, then $f : X \to Z$ is d-weakly continuous.*

Proof. Let $x \in X$ and let G be any neighborhood of $f(x)$ in Z. Then $V = G \cap Y$ is a neighborhood of $f(x)$ in Y. Since $f : X \to Y$ is d-weakly continuous at x, there is a neighborhood U of x such that $f(U) \subset \mathrm{cl}_{w_Y}(V)$. Since Y is open in Z, by Theorem 2.8 we have $\mathrm{cl}_{w_Y}(V) = \mathrm{cl}_Z(V) \cap Y$. Thus $f(U) \subset \mathrm{cl}_{w_Z}(V) \cap Y \subset \mathrm{cl}_{w_Z}(V) \subset \mathrm{cl}_{w_Z}(G)$ and so $f : X \to Z$ is d-weakly continuous at x. Hence the proof is complete. □

THEOREM 2.10. *Let $\{U_\alpha | \alpha \in I\}$ be an open covering of X and let f be a function from X to Y. If $f|_{U_\alpha} : U_\alpha \to Y$ is d-weakly continuous for any $\alpha \in I$, then f is d-weakly continuous.*

Proof. Let $x \in X$. Then since $\{U_\alpha | \alpha \in I\}$ is an open covering of X, there exists $\beta \in I$ such that $x \in U_\beta$. Let G be any neighborhood of $f(x)$. Then since $f|_{U_\beta} : U_\beta \to Y$ is d-weakly continuous at x, there is a neighborhood U of x in U_β such that $f|_{U_\beta}(U) = f(U) \subset \mathrm{cl}_w(G)$. Since U_β is open, U is open in X. Hence f is d-weakly continuous at x. This completes the proof of the theorem. □

THEOREM 2.11. *Let* $X = A \cup B$, *where* A *and* B *are disjoint closed subspaces in* X. *If* $f|_A : A \to Y$ *and* $f|_B : B \to Y$ *are d-weakly continuous, then* $f : X \to Y$ *is d-weakly continuous.*

Proof. Let $x \in X$. Since A and B are disjoint, the proof is divided into two the cases $x \in A$ and $x \in B$. First, let $x \in A$ and let G be any neighborhood of $f(x) = f|_A(x)$. Then since $f|_A$ is d-weakly continuous, there exists a neighborhood V of x in A such that $f|_A(V) = f(V) \subset \mathrm{cl}_w(G)$. Since A is a subspace of X, there is a neighborhood U of x in X such that $V = U \cap A$. Since B is closed, $U - B$ is a neighborhood of x in X. Since $U - B \subset X - B = A$, we have $U - B = (U - B) \cap A \subset U \cap A = V$. Thus we obtain $f(U - B) = f|_A(U - B) \subset f(V) \subset \mathrm{cl}_w(G)$. Therefore f is d-weakly continuous at x. Finally, let $x \in B$. Then in the same manner when $x \in A, f$ is d-weakly continuous at x. Hence the proof is complete. □

THEOREM 2.12. *Let* $f : X \to Y$ *and* $g : X \to Z$ *be d-weakly continuous and let* $h = f \times g : X \to Y \times Z$ *be a function defined by* $h(x) = (f(x), g(x))$. *Then* h *is d-weakly continuous.*

Proof. Let $x \in X$ and let G be any neighborhood of $h(x) = (f(x), g(x))$ in $Y \times Z$. Then by the definition of a product space, there are neighborhoods V of $f(x)$ in Y and W of $g(x)$ in Z such that $V \times W \subset G$. Since f and g are d-weakly continuous at x, there exist neighborhoods U_1, U_2 of x in X such that $f(U_1) \subset \mathrm{cl}_w(V)$ and $g(U_2) \subset \mathrm{cl}_w(W)$. Let $U = U_1 \cap U_2$. Then U is a neighborhood of x in X. Since

$$\begin{aligned} h(U) \subset f(U) \times g(U) \subset f(U_1) \times g(U_2) &\subset \mathrm{cl}_w(V) \times \mathrm{cl}_w(W) \\ &\subset \mathrm{cl}_w(V \times W) \subset \mathrm{cl}_w(G), \end{aligned}$$

h is d-weakly continuous at x. Hence the proof is complete. □

References

1. S.H. Choi, *The study on w-derived set and w-closure*, Master's thesis, Hanseo University, Seosan, Korea, 1999.
2. J. Dugundji, *Topology*, Allyn and Bacon Inc., Boston, 1967.
3. Y.H. Goo and D.H. Ry, *On the w-derived set*, J. Korea Soc. Math. Edu. Ser. B: Pure Appl. Math. **6** (1999), 13-16.
4. Y.H. Goo and Y.S. Ryu, *Properties of d-strongly continuous functions*, J. Korea Soc. Math. Edu. Ser. B: Pure Appl. Math. (to appear).
5. J. Husain, *Topology*, Plenum Press, New York and London, 1977.
6. H.K. Kim, *The characterization of H-closed spaces*, Master's thesis, Chunganm National University, Taejon, Korea, 1995.
7. S.H. Park, *Nearly continuous maps and weakly continuous maps*, Master's thesis, Chunganm National University, Taejon, Korea, 1999.

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