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THE LUSTERNIK-SCHNIRELMANN π_1-CATEGORY FOR A MAP

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Abstract. In this paper we shall define a concept of π_1-category for a map relative to a subset which is a generalization of both the category for a map and the π_1-category of a space, and study some properties of the π_1-category for a map relative to a subset.

1. Introduction

The concept of the category, $cat\ X$, of a space X was first devised by L. Lusternik and L. Schnirelmann [LS] in 1934 to finding the number of critical points of a smooth function on a smooth manifold X. The category, $cat\ X$, of X is the smallest number of sets, open and contractible in X, needed to cover X. In 1941, Fox [F] altered the origin definition by replacing closed sets by open sets in a covering appeared in the definition of $cat\ X$. In fact, these two notions coincide with each other for manifolds or more generally ANR spaces [J1]. The original notion of category can be generalized in a number of ways. One of them is the notion of the category, $cat\ f$, for a map $f : X \to Y$, due to Berstein and Ganea [BG] and another one is the notion of the π_1-category, $\pi_1 - cat\ X$, of a space X defined by Fox [F]. It is well known facts that $\pi_1 - cat\ X \leq cat\ X$ for any space X, and that $cat\ f \leq cat\ X$ and $cat\ f \leq cat\ Y$ for any map $f : X \to Y$.

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In this paper, we introduce the notion of the π_1-*category for a map* which is a generalization of both the category for a map and the π_1-category of a space, and study some properties of π_1-category for a map.

2. The Lusternik-Schnirelmann π_1-category for a map.

We define a concept of π_1-category for a map which is a generalization of the concept of π_1-category of a space, and also define a concept of π_1-category for a map relative to a subset and study some properties of a π_1-category for a map relative to a subset.

DEFINITION 2.1. Let $f : X \to Y$ be a map. A subset U of X is π_1-*contractible for a map* f if the restriction of f on U induces the trivial map $(f_{|U})_* = 0 : \pi_1(U, x) \to \pi_1(Y, f(x))$ for each $x \in U$. Then the Lusternik-Schnirelmann π_1-*category*, $\pi_1 - cat\ f$, *for* f is the least integer n such that X can be covered by the n open subsets $U_1, \cdots, U_n$ in X each of which is π_1-contractible for f. That is, $\pi_1 - cat\ f = min\{\#\{U_k\} | X \subset \bigcup U_k,\ U_k : open\ in\ X,\ \forall x \in U_k,\ (f_{|U_k})_* = 0 : \pi_1(U_k, x) \to \pi_1(Y, f(x))\}$. If no such number exists then $\pi_1 - cat\ f = \infty$.

DEFINITION 2.2. Let A be a subset of X and $f : X \to Y$ a map. Then the Lusternik-Schnirelmann π_1-*category*, $\pi_1 - cat\ f_A$, *for* f *relative to* A is the least integer n such that A can be covered by the n open subsets $U_1, \cdots, U_n$ in A each of which is π_1-contractible for $f_{|A}$. That is, $\pi_1 - cat\ f_A = min\{\#\{U_k\} | A \subset \bigcup U_k,\ U_k : open\ in\ A,\ \forall a \in U_k,\ (f_{|U_k})_* = 0 : \pi_1(U_k, a) \to \pi_1(Y, f(a))\}$. If no such number exists then $\pi_1 - cat\ f_A = \infty$.

Let $i_A : A \to X$ be the inclusion. Then $\pi_1 - cat\ i_A$ is called sometimes, $\pi_1 - cat_X A$, the π_1-*category of* A *in* X. Also it is clear that $\pi_1 - cat\ f_A = \pi_1 - cat\ (f \circ i_A) = \pi_1 - cat\ (f_{|A})$ and $\pi_1 - cat\ f_A = \pi_1 - cat\ f$ when $A = X$. The following proposition says that the category is a homotopy invariant.

PROPOSITION 2.3. *(1) If $f \sim g : X \to Y$, then $\pi_1 - cat\ f_A = \pi_1 - cat\ g_A$ for any subset A of X.*

(2) $\pi_1 - cat\ f_A = 1$ if and only if $(f_{|A})_* = 0 : \pi_1(A, a) \to \pi_1(Y, f(a))$ for each $a \in A$.

Proof. (1) Let U be an open subset of A. For any $a \in A$, there is a path ω in Y from $f(a)$ to $g(a)$ such that $h_\omega \circ (f_{|U})_* = (g_{|U})_* : \pi_1(U, a) \to \pi_1(Y, g(a))$, where $h_\omega : \pi_1(Y, f(a)) \to \pi_1(Y, g(a))$ is an isomorphism induced by the path ω. Thus we know that $(f_{|U})_* = 0$ iff $(g_{|U})_* = 0$ and $\pi_1 - cat\ f_A = \pi_1 - cat\ g_A$.
(2) Since $\{A\}$ is an open covering of A, it follows from the definition. □

The following theorem says that the category is subadditive.

THEOREM 2.4. *If $f : X \to Y$ is a map and $X = A_1 \cup A_2$, then $\pi_1 - cat\ f \leq \pi_1 - cat\ f_{A_1} + \pi_1 - cat\ f_{A_2}$, where $A_1,\ A_2$ are open subsets of X.*

Proof. Let $\pi_1 - cat\ f_{A_1} = m$ and $\pi_1 - cat\ f_{A_2} = n$. Then there exit a covering $\{U_i | U_i : \text{open in } A_1,\ (f_{|U_i})_* = 0 : \pi_1(U_i, a) \to (Y, f(a)),\ \forall a \in U_i,\ i = 1, \cdots, m\}$ of A_1 and a covering $\{V_j | V_j : \text{open in } A_2,\ (f_{|V_j})_* = 0 : \pi_1(V_j, b) \to (Y, f(b)),\ \forall b \in V_j,\ j = 1, \cdots, n\}$ of A_2. Now we want to show that $\{U_i,\ V_j |\ i = 1, \cdots, m,\ j = 1, \cdots, n\}$ is an open covering of X. Since U_i is open in A_1 and A_1 is open in X, U_i is open in X. Similarly, V_j is open in X. Since $A_1 = \cup_i U_i,\ A_2 = \cup_j V_j$ and $X = A_1 \cup A_2$, $\{U_i,\ V_j\}$ is an open covering of X. Hence $\pi_1 - cat\ f \leq cat\ f_{A_1} + \pi_1 - cat\ f_{A_2}$. □

THEOREM 2.5. *If $A \subset B \subset X$, then $\pi_1 - cat\ f_A \leq \pi_1 - cat\ f_B$.*

Proof. Let $\{U_\alpha\}$ be a covering of B such that U_α is open in B and for each $\alpha, x \in U_\alpha, (f_{|U_\alpha})_* = 0 : \pi_1(U_\alpha, x) \to \pi_1(Y, f(x))$. Then for each U_α, there exists an open V_α in X such that $U_\alpha = V_\alpha \cap B$. For each α, let $W_\alpha = U_\alpha \cap A$. Then $W_\alpha = U_\alpha \cap A = V_\alpha \cap B \cap A = V_\alpha \cap A$ is open in A. Since $A = B \cap A \subset \cup_\alpha (U_\alpha \cap A)$, $\{U_\alpha \cap A\}$ is an open covering of A. Also, for each α, consider the following commutative diagram;

$$\begin{array}{ccccccc} W_\alpha & \xrightarrow{i'_\alpha} & A & \xrightarrow{j} & B & \xrightarrow{i} & X \\ \| & & & & \uparrow{\scriptstyle i_\alpha} & & \\ W_\alpha & = & U_\alpha \cap A & \xrightarrow{k} & U_\alpha & & \end{array}$$

, where all maps are inclusions. For each W_α and each $x \in W_\alpha$, $(f_{|W_\alpha})_* = (f \circ i \circ j \circ i'_\alpha)_* = (f \circ i \circ i_\alpha \circ k)_* = (f_{|U_\alpha})_* \circ (k)_* = 0 : \pi_1(W_\alpha, x) \to \pi_1(Y, f(x))$. This proves the theorem. □

Taking $B = X$ in Theorem 2.5, we obtain the following corollary.

COROLLARY 2.6. *If A is a subset of X, then $\pi_1 - cat\ f_A \leq \pi_1 - cat\ f$.*

THEOREM 2.7. *Let $f : X \to Y$, $g : Y \to Z$ be maps and A a subset of X. Then $\pi_1 - cat\ (g \circ f)_A \leq min\ \{\pi_1 - cat\ f_A,\ \pi_1 - cat\ g_{f(A)}\}$.*

Proof. (1) We show that $\pi_1 - cat\ (g \circ f)_A \leq \pi_1 - cat\ f_A$. Let $\{U_\alpha\}$ be a covering of A such that for each α, U_α is open in A and for each $\alpha, x \in U_\alpha$, $(f_{|U_\alpha})_* = 0 : \pi_1(U_\alpha, x) \to \pi_1(Y, f(x))$. Since $(f_{|U_\alpha})_* = 0$, for each α, and each $x \in U_\alpha$, $((g \circ f)_{|U_\alpha})_* = g_*(f_{|U_\alpha})_* = 0 : \pi_1(U_\alpha, x) \to \pi_1(Z, g(f(x)))$. Thus $\pi_1 - cat\ (g \circ f)_A \leq \pi_1 - cat\ f_A$.
(2) We show that $\pi_1 - cat\ (g \circ f)_A \leq \pi_1 - cat\ g_{f(A)}$. Let $\{V_\alpha\}$ be a covering of $f(A)$ such that for each α, V_α is open in $f(A)$ and for each α and each $y \in V_\alpha$, $(g_{|V_\alpha})_* = 0 : \pi_1(V_\alpha, y) \to \pi_1(Z, g(y))$. Then for each α, there exists an open U_α in Y such that $V_\alpha = f(A) \cap U_\alpha$. Since $f : X \to Y$ is continuous, $f^{-1}(U_\alpha)$ is open in X and $f^{-1}(U_\alpha) \cap A$ is open in A. Moreover $A = f^{-1}(f(A)) \cap A = f^{-1}(\cup_\alpha V_\alpha) \cap A = \cup_\alpha (f^{-1}(V_\alpha) \cap A) = \cup_\alpha (f^{-1}(U_\alpha) \cap A)$. Therefore $\{f^{-1}(U_\alpha) \cap A\}$ is an open covering of A. For each α, consider the following commutative

diagram;

$$\begin{array}{ccccccc} V_\alpha & \xrightarrow{i_\alpha} & f(A) & \xrightarrow{i} & Y & \xrightarrow{g} & Z \\ \uparrow{\scriptstyle f} & & & & \uparrow{\scriptstyle f} & & \\ f^{-1}(U_\alpha)\cap A & & & \xrightarrow{j_\alpha} & X & & , \end{array}$$

where i_α, j_α, i are all inclusions. Then for each α and each $x \in f^{-1}(U_\alpha) \cap A$, $((g \circ f)_{|f^{-1}(U_\alpha)\cap A})_* = (g \circ f \circ j_\alpha)_* = (g \circ i \circ i_\alpha \circ f)_* = (g_{|V_\alpha})_* \circ (f_{|f^{-1}(U_\alpha)\cap A})_* = 0 : \pi_1(f^{-1}(U_\alpha) \cap A, x) \to \pi_1(Z, g(f(x)))$. Thus $\pi_1 - cat\ (g \circ f)_A \leq \pi_1 - cat\ g_{f(A)}$. From (1) and (2), we know that $\pi_1 - cat\ (g \circ f)_A \leq \min\ \{\pi_1 - cat\ f_A, \pi_1 - cat\ g_{f(A)}\}$. □

Corollary 2.8. *For any two maps $f : X \to Y$ and $g : Y \to Z$, $\pi_1 - cat\ (g \circ f) \leq \min\ \{\pi_1 - cat\ f, \pi_1 - cat\ g\}$. In particular, $\pi_1 - cat\ f \leq \pi_1 - cat\ X$ and $\pi_1 - cat\ f \leq \pi_1 - cat\ Y$.*

Corollary 2.9. *For any map $f : X \to Y$ and any subset A of X,$\pi_1 - cat\ f_A \leq \min\ \{\pi_1 - cat_Y f(A), \pi_1 - cat_X A\}$.*

Proof. In Theorem 2.7, take $g = 1_Y : Y \to Y$. Then we have that $\pi_1 - cat\ f_A \leq \pi_1 - cat\ 1_{f(A)} = \pi_1 - cat_Y f(A)$. From Corollary 2.8, we have that $\pi_1 - cat\ f_A = \pi_1 - cat\ (f \circ i_A) \leq \pi_1 - cat\ i_A = \pi_1 - cat_X A$, where $i_A : A \to X$ is the inclusion. Thus we have that $\pi_1 - cat\ f_A \leq \min\ \{\pi_1 - cat_Y f(A), \pi_1 - cat_X A\}$. □

Corollary 2.10. *If $h : X' \to X$ has a left homotopy inverse $k : X \to X'$, then for a subset A' of X', $\pi_1 - cat\ h_{A'} = \pi_1 - cat_{X'} A'$.*

Proof. From Corollary 2.8, we know that $\pi_1 - cat\ h_{A'} = \pi_1 - cat\ h \circ i' \leq \pi_1 - cat\ i' = \pi_1 - cat_{X'} A'$, where $i' : A' \to X'$ is the inclusion. Now we show that $\pi_1 - cat\ h_{A'} \geq \pi_1 - cat_{X'} A'$. Let $\{V_\alpha\}$ be a covering of A' such that for each α , V_α is open in A' and for each $x' \in V_\alpha$, $(h_{|V_\alpha})_* = 0 : \pi_1(V_\alpha, x') \to \pi_1(X, h(x'))$. Since $k \circ h \sim 1_{X'}$, we know that $(k \circ h)_{|V_\alpha} \sim i'_\alpha : V_\alpha \to X'$, where $i'_\alpha : V_\alpha \to X'$ is the inclusion. For any $x'_0 \in V_\alpha$, let $k \circ h(x'_0) = x'_1$. There exists a path ω in X' from

x_0' to x_1' such that $(k \circ h_{|V_\alpha})_* = \phi_\omega \circ (i_\alpha')_* \ : \pi_1(V_\alpha, x_0') \to \pi_1(X', x_1')$, where $\phi_\omega \ : \pi_1(X', x_0') \to \pi_1(X', x_1')$ is an isomorphism induced by the path ω. Since $(k \circ h_{|V_\alpha})_* = (k)_* \circ (h_{|V_\alpha})_* = 0$, we know that $(i_\alpha')_* = 0$. Therefore $\pi_1 - cat\ h_{A'} \geq \pi_1 - cat_{X'} A'$. This proves the corollary. □

The following theorem says that $\pi_1 - cat\ f_A$ is an invariant of homotopy type.

THEOREM 2.11. *Let $f : X \to Y$ be a map and $h : Y \to Z$ a homotopy equivalence with homotopy inverse $k : Z \to Y$. Then for any subset A of X, $\pi_1 - cat\ f_A = \pi_1 - cat\ (h \circ f)_A$.*

Proof. From Theorem 2.7, we know that $\pi_1 - cat\ (h \circ f)_A \pi_1 - cat\ f_A$. Thus we show that $\pi_1 - cat\ f_A \leq \pi_1 - cat\ (h \circ f)_A$. Let $\{V_\alpha\}$ be a covering of A such that for each α, V_α is open in A and for each $x \in V_\alpha$, $((h \circ f)_{|V_\alpha})_* = 0 : \pi_1(V_\alpha, x) \to \pi_1(Z, h(f(x)))$. Since $k \circ h \sim 1_Y$, we know $k \circ h \circ f \sim f : X \to Y$. For any $x_0 \in V_\alpha$, let $f(x_0) = y_0$ and $k(h(f(x_0))) = y_1$. Thus there exists a path ω in Y from y_0 to y_1 such that $((k \circ h \circ f)_{|V_\alpha})_* = \phi_\omega \circ (f_{|V_\alpha})_* \ : \pi_1(V_\alpha, x_0) \to \pi_1(Y, y_1)$, where $\phi_\omega : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is an isomorphism induced by the path ω. Since $((k \circ h \circ f)_{|V_\alpha})_* = (k)_* \circ ((h \circ f)_{|V_\alpha})_* = 0$, we have $(f_{|V_\alpha})_* = 0 \ : \pi_1(V_\alpha, x_0) \to \pi_1(Y, y_0)$. Thus we know that $\pi_1 - cat\ f_A \leq \pi_1 - cat\ (h \circ f)_A$. □

The following corollary says that $\pi_1 - cat\ f$ is an invariant of homotopy type.

COROLLARY 2.12. *Let $f : X \to Y$ be a map and $h : Y \to Z$ a homotopy equivalence Then $\pi_1 - cat\ f = \pi_1 - cat\ (h \circ f)$.*

THEOREM 2.13. *Let $f : X \to Y$ be a map and $h : X' \to X$ a homotopy equivalence with homotopy inverse $k : X \to X'$. If A' is a subset of X' with $k \circ h(A') \subset A'$, then $\pi_1 - cat\ (f \circ h)_{A'} = \pi_1 - cat\ f_{h(A')}$.*

Proof. From Theorem 2.7, we have that $\pi_1 - cat\ (f \circ h)_{A'} \leq \pi_1 - cat\ f_{h(A')}$. Thus we show that $\pi_1 - cat\ f_{h(A')} \leq \pi_1 - cat\ (f \circ h)_{A'}$.

Let $\{V_\alpha\}$ be a covering of A' such that for each α V_α is open in A' and for each $x' \in V_\alpha ((f \circ h)_{|V_\alpha})_* = 0 : \pi_1(V_\alpha, x') \to \pi_1(Y, f(h(x')))$. Then for each α, there exists an open set U_α in X' such that $V_\alpha = A' \cap U_\alpha$. Since $k : X \to X'$ is continuous, $k^{-1}(U_\alpha)$ is open in X and $k^{-1}(U_\alpha) \cap h(A')$ is open in $h(A')$. Thus we know, from the fact of $k \circ h(A') \subset A'$, that $k^{-1}(V_\alpha) \cap h(A') = k^{-1}(U_\alpha) \cap k^{-1}(A') \cap h(A') = k^{-1}(U_\alpha) \cap h(A')$ is open in $h(A')$. Since $\{V_\alpha\}$ is a covering of A' and $k \circ h(A') \subset A'$, we have $h(A') = k^{-1}(A') \cap h(A') \subset k^{-1}(\cup_\alpha V_\alpha) \cap h(A') = \cup_\alpha (k^{-1}(V_\alpha) \cap h(A'))$. Therefore $\{k^{-1}(V_\alpha) \cap h(A')\}$ is an open covering of $h(A')$. Since $h \circ k \sim 1_X$, we know $f \circ h \circ k \sim f : X \to Y$. For any $x_0 \in k^{-1}(V_\alpha) \cap h(A')$, let $f(x_0) = y_0$ and $f(h(k(x_0))) = y_1$. Thus there exists a path ω in Y from y_0 to y_1 such that $((f \circ h \circ k)_{|k^{-1}(V_\alpha) \cap h(A')})_* = \phi_\omega \circ (f_{|k^{-1}(V_\alpha) \cap h(A')})_* : \pi_1(k^{-1}(V_\alpha) \cap h(A'), x_0) \to \pi_1(Y, y_1)$, where $\phi_\omega : \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ is an isomorphism induced by the path ω. Since $((f \circ h \circ k)_{|k^{-1}(V_\alpha) \cap h(A')})_* = ((f \circ h)_{|V_\alpha})_* \circ (k_{|k^{-1}(V_\alpha) \cap h(A')})_* = 0$, we have $(f_{|k^{-1}(V_\alpha) \cap h(A')})_* = 0 : \pi_1(k^{-1}(V_\alpha) \cap h(A'), x_0) \to \pi_1(Y, y_0)$. Thus we know that $\pi_1 - cat\ f_{h(A')} \leq \pi_1 - cat\ (f \circ h)_{A'}$. □

In Theorem 2.13, taking $f = 1_X : X \to X$ and applying Corollary 2.10, we have the following corollary.

Corollary 2.14. *Let $h : X' \to X$ be a homotopy equivalence, A' a subset of X' and $h(A') = A$. Then $\pi_1 - cat_{X'} A' = \pi_1 - cat_X A$. In particular, $\pi_1 - cat\ X' = \pi_1 - cat\ X$.*

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