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ON A FUZZY BANACH SPACE

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Abstract. The main goal of this paper is to prove the following theorem ; Let (X, ρ_1) be a fuzzy normed linear space over K and (Y, ρ_2) be a fuzzy Banach space over K. If $\chi_{B_{\|\cdot\|}} \supseteq \rho_*$, then $(CF(X,Y), \rho_*)$ is a fuzzy Banach space, where $\rho_*(f) = \vee\{\theta \wedge \frac{1}{t(\theta,f)} \mid \theta \in (0,1)\}$, $f \in CF(X,Y)$, $B_{\|\cdot\|}$ is the closed unit ball on $(CF(X,Y), \|\cdot\|)$ and $\| f \| = \vee\{P^2_{\alpha^-}(f(x)) \mid P^1_{\alpha^-}(x) = 1, x \in X\}, f \in CF(X,Y), \alpha \in (0,1)$.

1. Introduction

The notions of fuzzy seminorm and fuzzy norm have introduced by Katsaras[2]. And in [5], Krishna and Sarma has shown the properties of fuzzy norms on the set of all fuzzy continuous linear maps those defined on fuzzy normed linear space to other fuzzy normed linear space. Rhie, Choi and Kim [7] introduced the notions of the fuzzy α-Cauchy sequence and fuzzy completeness, and in [8], Rhie and Hwang studied the relation between fuzzy seminorms and ordinary seminorms generated by the fuzzy seminorms.

In this paper, we show that

$$\| f \| = \vee\{P^2_{\alpha^-}(f(x)) \mid P^1_{\alpha^-}(x) = 1, x \in X\}$$

is a norm on the linear space of all fuzzy continuous linear maps, and prove that this space containing several conditions is a fuzzy Banach space.

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2. Preliminaries

In this section, we explain some basic definitions and results.

Let X be a nonempty set. A fuzzy set in X is an element of the set I^X of all functions from X into the unit interval I. χ_A denotes the characteristic function of the set A. $\vee$ and $\wedge$ are used for the supremum and infimum of the family respectively, $R_+ = [0, \infty)$.

Definition 2.1. [**3**] *Let X be a vector space. For $\mu \in I^X$ and t a scalar, the fuzzy set $t\mu$ is the image of μ under the map $g : X \to X$, $g(x) = tx$, that is if $\mu \in I^X$ and $t \in K$, then*

$$(t\mu)(x) = \begin{cases} \mu(x/t) & \text{if} \quad t \neq 0 \\ 0 & \text{if} \quad t = 0 \ \text{ and } \ x \neq 0 \\ \vee\{\mu(y) \mid \ y \in X\} & \text{if} \quad t = 0 \ \text{ and } \ x = 0. \end{cases}$$

Let α be a fuzzy set in the scalar field K and μ a fuzzy set in a vector space X. Let $\alpha \times \mu$ be the fuzzy subset in $I^{K \times X}$ and the map $h : K \times X \to X$, $h(t, x) = tx$. Then we denote the image of $\alpha \times \mu$ under h by $\alpha\mu$. That is

$$\alpha\mu(y) \ = \ h(\alpha \times \mu)(y) \ = \ \vee\{(\alpha \times \mu)(t, x) \mid (t, x) \in h^{-1}(y)\}.$$

Definition 2.2. [**3**] *$\mu \in I^X$ is said to be*

1. *convex* if $t\mu + (1 - t)\mu \subseteq \mu$ *for each* $t \in [0, 1]$
2. *balanced* if $t\mu \subseteq \mu$ *for each* $t \in K$ *with* $|\, t \,| \leq 1$
3. *absolutely convex* if μ *is convex and balanced*
4. *absorbing* if $\vee\{t\mu(x) \mid t > 0\} = 1$ *for all* $x \in X$.

Definition 2.3. [**2**] *A fuzzy seminorm on X is a fuzzy set ρ in X which is absolutely convex and absorbing. If in addition $\wedge\{(t\rho)(x) \mid t > 0\} = 0$ for $x \neq 0$, then ρ is called a fuzzy norm.*

Definition 2.4. [**4**] *If ρ is a fuzzy seminorm on X, then for every $\epsilon \in (0, 1)$, $P_\epsilon : X \to R_+$ is defined by*

$$P_\epsilon(x) = \wedge\{t > 0 \mid t\rho(x) > \epsilon\}$$

and for every $x \in X$, $P_{\alpha^-} : X \to R_+$ *is also defined by*

$$P_{\alpha^-}(x) = \vee\{P_\epsilon(x) \mid \epsilon < \alpha\}.$$

DEFINITION 2.5. **[4]** *The* P_ϵ *in Definition 2.4 is a seminorm on* X*. Further* P_ϵ *is a norm on* X *for each* $\epsilon \in (0,1)$ *if and only if* ρ *is a fuzzy norm on* X*.*

DEFINITION 2.6. **[5]** *A fuzzy set* $\mu \in I^X$ *is called a fuzzy point iff*

$$\mu(z) = \begin{cases} \alpha & \text{if } z = x, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha \in (0,1)$*. We denote this fuzzy point with support* x *and value* α *by* (x, α)*.*

3. Main results

In this section, we obtain some properties of the linear space of all fuzzy continuous linear maps.

DEFINITION 3.1. **[7]** *Let* $\alpha \in (0,1)$*. A sequence of fuzzy points* $\{\mu_n = (x_n, \alpha_n)\}$ *is said to be a fuzzy* α*-Cauchy sequence in a fuzzy normed linear space* (X, ρ) *if for each neighborhood* N *of* 0 *with* $N(0) > \alpha$*, there exists a positive integer* M *such that* $n, m \geq M$ *implies* $\mu_n - \mu_m = (x_n - x_m, \alpha_n \wedge \alpha_m) \leq N$*. A fuzzy normed linear space* (X, ρ) *is said to be fuzzy* α*-complete if every fuzzy* α*-Cauchy sequence* $\{\mu_n\}$ *converges to a fuzzy point* $\mu = (x, \alpha)$ *(refer to Definition 2.13 of [5]).* (X, ρ) *is said to be fuzzy complete if it is fuzzy* α*-complete for every* $\alpha \in (0,1)$*. A fuzzy complete fuzzy normed linear space is said to be a fuzzy Banach space.*

THEOREM 3.2. **[4]** *The* P_{α^-} *in Definition 2.4 is a seminorm on* X*. Further* P_{α^-} *is a norm on* X *for each* $\alpha \in (0,1)$ *if and only if* ρ *is a fuzzy norm on* X*.*

LEMMA 3.3. *For each* ϵ *in* $(0,1)$*, if* f *is continuous on* (X, P_ϵ^1) *to* (Y, P_ϵ^2) *where* P_ϵ^1 *and* P_ϵ^2 *are norms. Then for each* α *in* $(0,1)$*,* f :

$(X, P^1_{\alpha^-}) \to (Y, P^2_{\alpha^-})$ *is continuous, where* $P^1_{\alpha^-}(x) = \vee\{P^1_\epsilon(x) \mid \epsilon < \alpha\}$ *and* $P^2_{\alpha^-}(f(x)) = \vee\{P^2_\epsilon(f(x)) \mid \epsilon < \alpha\}$.

Proof. Since for each ϵ in $(0,1)$, $f : (X, P^1_\epsilon) \to (Y, P^2_\epsilon)$ is continuous, for each $\epsilon \in (0,1)$ and any $x_0 \in X$, and positive real number t, there is a positive real number η such that $P^1_\epsilon(x - x_0) < \eta$ implies

$$P^2_\epsilon(f(x) - f(x_0)) < \frac{t}{2}.$$

From $P^1_\epsilon(x - x_0) < \eta$, there is a positive real number δ such that

$$P^1_\epsilon(x - x_0) \le \delta < \eta.$$

If α in $(0,1)$ is fixed, then

$$\begin{aligned} & P^1_{\alpha^-}(x - x_0) < \delta \\ \Rightarrow\quad & P^1_\epsilon(x - x_0) \le \delta < \eta \ \text{ for each } \ \epsilon < \alpha \\ \Rightarrow\quad & P^2_\epsilon(f(x) - f(x_0)) < \frac{t}{2} \ \text{ for each } \ \epsilon < \alpha \\ \Rightarrow\quad & \vee\{P^2_\epsilon(f(x) - f(x_0)) \mid \epsilon < \alpha\} \le \frac{t}{2} < t, \end{aligned}$$

and so, for each $\alpha \in (0,1)$ and any $x_0 \in X$ there is a positive real number δ such that $P^1_{\alpha^-}(x - x_0) < \delta$ implies $P^2_{\alpha^-}(f(x) - f(x_0)) < t$. □

THEOREM 3.4. [**5**] *Let* (X, ρ_1), (Y, ρ_2) *be fuzzy normed linear spaces and* $f : (X, \rho_1) \to (Y, \rho_2)$ *be continuous. Then for each* $\epsilon \in (0,1)$, $f : (X, P^1_\epsilon) \to (Y, P^2_\epsilon)$ *is continuous.*

LEMMA 3.5. *For a fixed* $\alpha \in (0,1)$, *if we define a function* $\|\cdot\|$ *on* $CF((X, \rho_1), (Y, \rho_2))$ *by*

$$\| f \| = \vee\{P^2_{\alpha^-}(f(x)) \mid P^1_{\alpha^-}(x) = 1, x \in X\}.$$

Then $\|\cdot\|$ *is a norm on* $CF((X, \rho_1), (Y, \rho_2))$.

Proof. i) For $f, g \in CF((X,\rho_1),(Y,\rho_2))$,

$$\begin{aligned}\| f+g \| &= \vee\{P^2_{\alpha^-}(f+g)(x) \mid P^1_{\alpha^-}(x)=1, x\in X\}\\ &= \vee\{P^2_{\alpha^-}(f(x)+g(x)) \mid P^1_{\alpha^-}(x)=1, x\in X\}\\ &\leq \vee\{P^2_{\alpha^-}(f(x))+P^2_{\alpha^-}(g(x)) \mid P^1_{\alpha^-}(x)=1\}\\ &\leq \vee\{\| f \| P^1_{\alpha^-}(x)+ \| g \| P^1_{\alpha^-}(x) \mid P^1_{\alpha^-}(x)=1\}\\ &= \vee\{(\| f \| + \| g \|)P^1_{\alpha^-}(x) \mid P^1_{\alpha^-}(x)=1\}\\ &= \| f \| + \| g \| .\end{aligned}$$

ii) For $f \in CF((X,\rho_1),(Y,\rho_2))$, if $\| f \|= 0$, then since $P^2_{\alpha^-}$ is a crip norm and since f is continuous, for any $x \in X$,

$$0 \leq P^2_{\alpha^-}(f(x)) \leq \| f \| P^1_{\alpha^-}(x),$$

and so, $P^2_{\alpha^-}(f(x)) = 0$. Hence for any $x \in X$, $f(x) = 0$.

iii) For any $a \in K$ and $f \in CF((X,\rho_1),(Y,\rho_2))$,

$$\begin{aligned}\| af \| &= \vee\{P^2_{\alpha^-}((af)(x)) \mid P^1_{\alpha^-}(x)=1, x\in X\}\\ &= \vee\{P^2_{\alpha^-}(af(x)) \mid P^1_{\alpha^-}(x)=1, x\in X\}\\ &= \vee\{\mid a \mid P^2_{\alpha^-}(f(x)) \mid P^1_{\alpha^-}(x)=1\}\\ &= \mid a \mid \| f \| .\end{aligned}$$

□

LEMMA 3.6. *Let ρ_1, ρ_2 be two fuzzy seminorms on a linear space X. If for every $x \in X$, $\rho_1(x) \leq \rho_2(x)$, then for every $\epsilon \in (0,1)$, $P^1_\epsilon(x) \geq P^2_\epsilon(x)$ for all $x \in X$.*

Proof. Since $\rho_1(x) \leq \rho_2(x)$ for all $x \in X$,

$$t\rho_1(x) \leq t\rho_2(x)$$

for all $t \in K$, $x \in X$. Let $\epsilon \in (0,1)$ and $x \in X$ be fixed. Since $t\rho_1(x) > \epsilon$ implies $t\rho_2(x) > \epsilon, \{t > 0 \mid t\rho_1(x) > \epsilon\}$ is a subset of

$\{t' > 0 \mid t'\rho_2(x) > \epsilon\}$. Hence

$$\wedge\{t > 0 \mid t\rho_1(x) > \epsilon\} \geq \wedge\{t' > 0 \mid t'\rho_2(x) > \epsilon\},$$

equivalently $P^1_\epsilon(x) \geq P^2_\epsilon(x)$. This completes the proof. □

Definition 3.7. [5] *Let* (X,ρ_1), (Y,ρ_2) *be fuzzy normed linear spaces and* $CF(X,Y)$ *be the linear space of all fuzzy continuous linear maps from* (X,ρ_1) *to* (Y,ρ_2). *For each* $\theta \in (0,1)$, $t_\theta : CF(X,Y) \to R_+$ *is defined by*

$$t_\theta(f) = \wedge\{s > 0 \mid \rho_2(f(x)) \geq \theta \wedge \rho_1(sx) \text{ for all } x \in X\}.$$

We write $t_\theta(f) = t(\theta, f)$. *And the fuzzy norm* $\rho_* : CF(X,Y) \to [0,1]$ *is defined by* $\rho_*(f) = \vee_{\theta\in(0,1)}\{\theta \wedge 1/[t(\theta,f)]\}$, *for any* $f \in CF(X,Y)$.

Theorem 3.8. *Let* (X,ρ_1) *be a fuzzy normed linear space over* K *and* (Y,ρ_2) *be a fuzzy Banach space over* K. *If* $\chi_{B_{\|\cdot\|}} \supseteq \rho_*$, *then* $(CF(X,Y),\rho_*)$ *is a fuzzy Banach space, where* $\rho_*(f) = \vee\{\theta \wedge \frac{1}{t(\theta,f)} \mid \theta \in (0,1)\}$, $f \in CF(X,Y)$, $B_{\|\cdot\|}$ *is the closed unit ball on* $(CF(X,Y), \|\cdot\|)$ *and* $\| f \| = \vee\{P^2_{\alpha^-}(f(x)) \mid P^1_{\alpha^-}(x) = 1, x \in X\}, f \in CF(X,Y), \alpha \in (0,1)$.

Proof. α-Cauchy sequence in $CF(X,Y)$. Then by [7, Theorem 3.2], for each $t > 0$, there is an M in Z^+ such that if $n,m \geq M$, then $\alpha_n \wedge \alpha_m \leq \alpha$ and $P^*_{(\alpha_n\wedge\alpha_m)^-}(f_n - f_m) < t$. Let $\chi_{B_{\|\cdot\|}} = \rho_0$. Then for each $\beta \in (0,1)$, $\| f \| = P^0_{\beta^-}(f)$ where

$$\begin{aligned} P^0_{\beta^-}(f) &= \vee\{P^0_\epsilon(f) \mid \epsilon < \beta\} \\ &= \vee\{\wedge\{t > 0 \mid t\rho_0(f) > \epsilon\} \mid \epsilon < \beta\}. \end{aligned}$$

Hence for $\alpha_n \wedge \alpha_m \leq \alpha$,

$$\begin{aligned} P^0_{(\alpha_n \wedge \alpha_m)^-}&(f_n - f_m) \\ &= \| f_n - f_m \| \\ &= \vee \{P^2_{(\alpha_n \wedge \alpha_m)^-}(f_n - f_m)(x) \mid P^1_{(\alpha_n \wedge \alpha_m)^-}(x) = 1, x \in X\} \\ &= \vee \{P^2_{(\alpha_n \wedge \alpha_m)^-}(f_n(x) - f_m(x)) \mid P^1_{(\alpha_n \wedge \alpha_m)^-}(x) = 1\}, \end{aligned}$$

and so, for any $x \in X$,

$$P^2_{(\alpha_n \wedge \alpha_m)^-}(f_n(x) - f_m(x)) \leq P^0_{(\alpha_n \wedge \alpha_m)^-}(f_n - f_m) \cdot P^1_{(\alpha_n \wedge \alpha_m)^-}(x).$$

And since $\chi_{B_{\|\cdot\|}} \supseteq \rho_*$,

$$P^0_{\gamma^-}(f) \leq P^*_{\gamma^-}(f),$$

$\gamma \in (0,1),\ f \in CF(X,Y)$. Therefore for any $x \in X$,

$$\begin{aligned} P^2_{(\alpha_n \wedge \alpha_m)^-}(f_n(x) - f_m(x)) &\leq P^0_{(\alpha_n \wedge \alpha_m)^-}(f_n - f_m) \cdot P^1_{(\alpha_n \wedge \alpha_m)^-}(x) \\ &\leq P^*_{(\alpha_n \wedge \alpha_m)^-}(f_n - f_m) \cdot P^1_{(\alpha_n \wedge \alpha_m)^-}(x) \\ &< t \cdot P^1_{(\alpha_n \wedge \alpha_m)^-}(x). \end{aligned}$$

Thus $\{f_n(x)\}$ is an α-Cauchy sequence in (Y, ρ_2) for each $x \in X$. Since (Y, ρ_2) is a fuzzy Banach space, $f_n(x)$ converges to $f(x)$ for each $x \in X$, so (f_n, α_n) is convergent to (f, α). Therefore $(CF(X,Y), \rho_*)$ is a fuzzy Banach space. □

COROLLARY 3.9. *If (X, ρ) is a fuzzy normed linear space over K, $\chi_{B_{\|\cdot\|}} \supseteq \rho_*$ and for $f \in CF(X,K)$*

$$\| f \| = \vee\{| f(x) | \mid P_{\alpha^-}(x) = 1, x \in X\}.$$

Then $(CF(X,K), \rho_*)$ is a fuzzy Banach space over K.

Proof. Since (X, ρ) and $(K, \chi_{B_{|\cdot|}})$ are fuzzy normed linear spaces, $(CF(X,K), \rho_*)$ is a fuzzy normed linear space over K. And since

$(K, \chi_{B_{|\cdot|}})$ is a fuzzy Banach space, $(CF(X,K), \rho_*)$ is a fuzzy Banach space over K. □

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