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A SIMPLE PROOF OF GRÖTZSCH'S PRINCIPLE

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ABSTRACT. The purpose of this paper is to apply the extremal length to conformal mappings. We drive an interesting formula of Grötzsch by an alternative simple method.

1. Introduction

The method of extremal length is a useful tool in a wide variety of areas. Especially, it has been successfully applied to conformal mappings, analytic functions of a complex variable. Extremal length was introduced as a conformally invariant measure of curve families. This development appeared in Ahlfors and Beurling [3].

Throughout this paper $\mathbb{C}$ will denote the finite complex plane, D is a domain in $\mathbb{C}$.

2. Extremal length

Let Γ be a family whose elements γ are locally rectifiable curves in D. We shall introduce a geometric quantity $\lambda(\Gamma)$, called the extremal length of Γ.

Let $\rho(z)$ be a non-negative Borel measurable function. Every curve γ has a well-defined "ρ - length"

$$L(\gamma, \rho) = \int_\gamma \rho(z)\,|dz|, \tag{1}$$

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which may be infinite, and D has a "ρ - area"

$$A(D,\rho) = \iint_D [\rho(z)]^2\,dx\,dy. \tag{2}$$

In order to define an invariant which depends on the whole set Γ, we introduce the minimum length

$$L(\Gamma,\rho) = \inf_{\gamma\in\Gamma} L(\gamma,\rho), \tag{3}$$

where we agree that $L(\Gamma,\rho)=\infty$ in the case Γ is empty.

DEFINITION 2.1.[1] The *extremal length* of Γ in D is defined as

$$\lambda(\Gamma) = \lambda_D(\Gamma) = \sup_{\rho} \frac{[L(\Gamma,\rho)]^2}{A(D,\rho)}, \tag{4}$$

where ρ is subject to the condition $0 < A(D,\rho) < \infty$, obviously $0 \le \lambda(\Gamma) \le \infty$.

REMARK 1. (i) $\lambda_D(\Gamma)$ depends only on Γ and not on D. Accordingly, we shall henceforth simplify the notation to $\lambda(\Gamma)$ ([1]).

(ii) Since almost every curve in $\mathbb{C}$ is rectifiable, the non-rectifiable curves of a family Γ have no influence on the extremal length of Γ.So, let us use curve or arc instead of locally rectifiable curve ([8]).

There are two special cases in which the extremal length is very easy to determine explicitly.

PROPOSITION 2.2.[2] *Let S be a rectangle of sides a and b. Let Γ be the family of arcs in S which join the sides of length b. Then*

$$\lambda(\Gamma) = \frac{a}{b}.$$

PROPOSITION 2.3.[5] *Let E be the annulus $E = \{z \mid a < |z| < b\}$. Let Γ be the family of arcs in E which join the two contours. Then*

$$\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{b}{a} \ .$$

The conformal invariance of extremal length is an immediate consequence of the definition.

PROPOSITION 2.4.[7](CONFORMAL INVARIANCE OF EXTREMAL LENGTH) *Let $z^* = f(z)$ be a 1-1 conformal mapping on D upon a domain D^* and Γ be a family of curves in D, then*

$$\lambda(\Gamma) = \lambda[f(\Gamma)].$$

PROPOSITION 2.5.[7] *Suppose there exist disjoint open sets G_n containing the curves in Γ_n. If $\cup_n \Gamma_n \subset \Gamma$, then*

$$\sum_n \frac{1}{\lambda(\Gamma_n)} \ \leq \ \frac{1}{\lambda(\Gamma)} \ .$$

3. Geometric applications of extremal length

Here we shall give an alternative simple proof of the well-known result by the method of extremal length.

GRÖTZSCH'S PRINCIPLE. [4] *Suppose that we have a set of n disjoint general quadrilaterals Q_k, for $k = 1, 2, \ldots, n$, that are contained in the annulus $\Delta = \{z \mid \ r < |z| < R\}$, $(0 < r < R,\ R \neq \infty)$ and that are bounded by Jordan curves each of which has an arc, in common with each of the circles $\{z \mid |z| = r\}$ and $\{z \mid |z| = R\}$. (The Q_k can be regarded as strips extending from the inner to the outer circle.)*

If these domains Q_k are mapped onto rectangles S_k with sides equal respectively to a_k and b_k in such a way that the arcs referred to are mapped into sides of lengths a_k, then

$$\sum_{k=1}^{n} \frac{a_k}{b_k} \leq \frac{2\pi}{\log \frac{R}{r}}, \tag{5}$$

with equality holding only if the Q_k are domains of the form $\{(z, \arg z)|\ r < |z| < R,\ \phi_k < \arg z < \phi_{k+1}\}$ completely filling the annulus.

Original Proof. Let us cut the annulus $\Delta = \{z \mid r < |z| < R\}$ along an arc belonging to the boundary one of the strips Q_k and let us map the simply connected domain obtained onto the plane $t = \log z = u + iv$. This mapping maps the strips Q_k into strips Q'_k that extend from the line $u = \log r$ to the line $u = \log R$. Let $z = g_k(\zeta)$ denote the functions inverse to those mentioned in this principle. These functions map the strips Q_k onto the rectangles

$$B_k = \{(\xi, \eta)|\ 0 < \xi < b_k,\ 0 < \eta < a_k\}.$$

Then the functions $t = \log g_k(\zeta) = f_k(\zeta)$ map B_k onto the strips Q'_k. The images of the sides of length a_k under this mapping lie on the straight lines $u = \log r$ and $u = \log R$.

We denote by I_k the area of the strip Q'_k. Then, we obviously have

$$\sum_{k=1}^{r} I_k \leq 2\pi \log \frac{R}{r}.$$

On the other hand,

$$I_k = \iint_{B_k} |f'_k(\zeta)|^2 d\xi d\eta = \int_0^{a_k} \left(\int_0^{b_k} |f'_k(\zeta)|^2 d\xi \right) d\eta$$
$$\geq \int_0^{a_k} \frac{1}{b_k} \left(\int_0^{b_k} |f'_k(\zeta)| d\xi \right)^2 d\eta,$$

the inequality being obtained on the basis of the Schwarz inequality. But the integrals

$$\int_0^{b_k} |f_k'(\zeta)|d\xi$$

are the lengths of the images of the sides parallel to the sides of length b_k; that is, they are the lengths of certain curves extending from the line $u = \log r$ to the line $u = \log R$. Therefore

$$\int_0^{b_k} f_k'(\zeta)d\xi \geq \log \frac{R}{r},$$

and, consequently,

$$I_k \geq (\frac{a_k}{b_k})(\log \frac{R}{r})^2.$$

If we substitute this into

$$\sum_{k=1}^{n} I_k \leq 2\pi \log \frac{R}{r}.$$

and divide by $(\log \frac{R}{r})^2$, we obtain (5).

For equality to hold in (5), it is obviously necessary that the strips Q_k completely fill the annulus $\Delta = \{z : r < |z| < R\}$ and that the images (referred to above) of all the lines parallel to the sides of length b_k be lines parallel to the real axis, that is, that the Q_k be of the form $\{(z, \arg z) : r < |z| < R, \phi_k < \arg z < \phi_{k+1}\}$. A simple check shows that in this case equality does indeed hold in (5). □

Alternative Simple Proof. We can map an arbitrary general quadrilateral conformally onto a rectangle,(ref. [6]). Let $w = h_k(z)$ be a 1-1 conformal mappings on Q_k upon S_k respectively. Let Γ be the family of arcs in Δ which join the two boundary circles, and let Γ_k be the family of arcs in Q_k which join the two sides of $Q_k \subset \partial\Delta$. Then by Proposition 2.4 and Proposition 2.2,

$$\lambda(\Gamma_k) = \lambda[h_k(\Gamma_k)] = \frac{b_k}{a_k} \ . \tag{6}$$

By the hypothesis, there exist disjoint open sets $Q_k (k = 1, 2, \ldots , n)$ containing Γ_k and $\cup_k \Gamma_k \subset \Gamma$. Hence by Proposition 2.5,

$$\sum_{k=1}^{n} \frac{1}{\lambda(\Gamma_k)} \ \leq \ \frac{1}{\lambda(\Gamma)} \ . \tag{7}$$

Therefore by Proposition 2.3, (6) and (7), we obtain (5). The proof is complete. □

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