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# TOPOLOGICAL STABILITY OF INVERSE SHADOWING SYSTEMS

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Abstract. The inverse shadowing property of a dynamical system is an "inverse" form of the shadowing property of the system. Recently, Kloeden and Ombach proved that if an expansive system on a compact manifold has the shadowing property then it has the inverse shadowing property. In this paper, we study topological stability of the inverse shadowing dynamical systems. In particular, we show that if an expansive system on a compact manifold has the inverse shadowing property then it is topologically stable, and so it has the shadowing property.

Let $X$ be a compact metric space with a metric $d$, and let $f$ be a homeomorphism (or dynamical system) mapping $X$ onto itself.

Definition 1. A *$\delta$-pseudo orbit* of $f$ is a sequence of points $\xi = \{x_i \in X : i \in \mathbb{Z}\}$ such that

$$d(f(x_i), x_{i+1}) < \delta, \quad i \in \mathbb{Z}$$

The notion of a pseudo orbit plays an important role in the general qualitative of dyamical systems. It is used to define some types of invariant sets such as the chain recurrent set or chain prolongation sets([5]).

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Definition 2. We say that a pseudo orbit $\xi = \{x_i \in X : i \in \mathbb{Z}\}$ is $\varepsilon$ - *shadowed* by a point $x \in X$ if the inequality

$$d(f^i(x), x_i) < \varepsilon, \quad i \in \mathbb{Z}$$

holds.

Definition 3. A homeomorphism $f$ is said to have the *shadowing property* if given $\varepsilon > 0$ there exists $\delta > 0$ such that any $\delta$ -pseudo orbit of $f$ is $\varepsilon$-shadowed by a point in $X$. If any such $\delta$ -pseudo orbit of $f$ is $\varepsilon$-shadowed by not more than one point $x \in X$, than we say that $f$ has the *shadowing uniqueness property* (SUP)

Thus the existence of a shadowing point for a pseudo orbit $\xi$ means that $\xi$ is close to a real orbit of $f$. The theory of shadowing was developed intensively in recent years and became a significant part of the qualitative theory of dynamical systems containing a lot of interesting and deep results([4],[5],[6]).

In this paper we consider the inverse shadowing property of a dynamical system which is an "inverse" form of the shadowing property of a dynamical system ([3]).

Let $M$ be a smooth $n$-dimensional closed (i.e., compact and boundaryless) manifold, and $d$ be a Riemannian metirc on $M$. We consider the space $Z(M)$ of homeomorphisms on $M$ with the metric $d_0$ defined by the formula

$$d_0(f, g) = \max\{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)) : x \in M\}.$$

Definition 4. A homeomorphism $f \in Z(M)$ is *expansive* if there exist a constant $e > 0$ such that $d(f^n(x), f^n(y)) \leq e$ for all $n \in \mathbb{Z}$ implies that $x = y$. Such a number $e > 0$ is called an *expansive constant* of $f$.

For any homeomorphism $g \in Z(M)$, we define $e(g)$ by

$$e(g) = \sup\{e \geq 0 : d(g^n(x), g^n(y)) \leq e \text{ for all } n \in \mathbb{Z} \text{ implies } x = y\}.$$

DEFINITION 5. A homeomorphism $f \in Z(M)$ is *strongly expansive* if the map $e : Z(M) \to [0, \infty)$ given by $g \longmapsto e(g)$ is lower semicontinuous in the $C^0$-topology at $f$; that is, for any $0 < \varepsilon < e(f)$ there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ then $\varepsilon$ is an expansive constant of $g$.

DEFINITION 6. Let $M^{\mathbb{Z}}$ be the compact space (with the product topology) of all two sided sequences $\xi = \{x_n : n \in \mathbb{Z}\}$ with components $x_n \in M$. For $\delta > 0$, we let $\Phi_f(\delta) \subset M^{\mathbb{Z}}$ be the set of all $\delta$-pseudo orbits of $f$. A mapping $\varphi : M \to \Phi_f(\delta) \subset M^{\mathbb{Z}}$ is said to be a $\delta$-*method* for $f$. Then each $\varphi(x) \in \Phi_f(\delta)$ is a $\delta$-pseudo orbit of $f$. We say that $\varphi$ is a *continuous* $\delta$-*method* for $f$ if $\varphi$ is continuous.

For convenience, we denote $\varphi(x)$ by

$$\{\varphi(x)_n\}_{n \in \mathbb{Z}}.$$

The set of all $\delta$-methods (resp. continuous $\delta$-methods) for $f$ will be denoted by $\mathcal{T}_0(f, \delta)$ (resp. $\mathcal{T}_c(f, \delta)$), and we define $\mathcal{T}_0(f)$ and $\mathcal{T}_c(f)$ by

$$\mathcal{T}_0(f) = \bigcup_{\delta > 0} \mathcal{T}_0(f, \delta) \quad \text{and} \quad \mathcal{T}_c(f) = \bigcup_{\delta > 0} \mathcal{T}_c(f, \delta),$$

respectively. Moreover, if $g : M \to M$ is a homeomorphism then $g$ induces a continuous method $\varphi_g : M \to M^{\mathbb{Z}}$ for $f$ by defining

$$\varphi_g(x) = \text{the orbit of } g \text{ through } x.$$

We can easily see that if $d_0(f, g) < \delta$ then $\varphi_g$ is a continuous $\delta$-method for $f$. Denote $\mathcal{T}_h(f)$ by the set of all continuous methods for $f$ which are induced by homeomorpisms on $M$. Then we have the following inclusions :

$$\mathcal{T}_h(f) \subset \mathcal{T}_c(f) \subset \mathcal{T}_0(f).$$

DEFINITION 7. A homeomorphism $f : M \to M$ is said to have the *inverse shadowing property* (with respect to the class $\mathcal{T}_h(f)$) if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $\delta$-method $\Phi$ in $\mathcal{T}_h(f)$ and any point $y \in M$ there exists a point $x \in M$ for which

$$d(f^n(y), \Phi(x)_n) < \varepsilon, \quad \text{for all} \quad n \in \mathbb{Z},$$

If for any $\delta$-method $\Phi$ in $\mathcal{T}_h(f)$ and any point $y \in M$ there is not more than one point $x \in M$ for which

$$d(f^n(y), \Phi(x)_n) < \varepsilon, \quad \text{for all} \quad n \in \mathbb{Z},$$

then we say that $f$ has the *inverse shadowing uniqueness property*(IS UP)(with respect to the class $\mathcal{T}_h(f)$).

In numerical calculation, an inverse form of the shadowing concept is interesting and many results are obtained. Recently, Corless and Pilyugin proved that every diffeomorphism $f \in Z(M)$ with the strong tranversality condition does not have the inverse shadowing property with respect to the class $\mathcal{T}_0(f, \delta)$([1]). Kloeden and Ombach proved that every Anosov diffeomorphism has the inverse shadowing property with respect to the class $\mathcal{T}_c(f, \delta)$([2]). Lee and Choi proved that every Morse-Smale diffeomorphism on $M$ has the inverse shadowing property with respect to the class $\mathcal{T}_h(f, \delta)$ but it does not have the inverse shadowing property with respect to the class $\mathcal{T}_c(f, \delta)$([3]).

Throughout the paper, "the inverse shadowing property" implies "the inverse shadowing property with respect to the class $\mathcal{T}_h(f, \delta)$".

DEFINITION 8. We say that a homeomorphism $f \in Z(M)$ is *topologically stable* if given $\varepsilon > 0$ there exists a neighborhood $W$ of $f$ in $Z(M)$ such that for any homeomorphism $g \in W$ there is a continuous mapping $h$ of $M$ onto $M$ having the following properties :

(1) $d(x, h(x)) < \varepsilon$ for $x \in M$,

(2) $f \circ h = h \circ g$,

THEOREM 9. *Any homeomorphism $f : M \to M$ with the ISUP is topologically stable.*

*Proof.* We can easily check that if $f$ has the ISUP then it is expansive. Let $e > 0$ be an expansive constant of $f$ and $\varepsilon > 0$ a constant with $\varepsilon < \frac{e}{12}$. Since $f$ has the ISUP, we can choose $\delta > 0$ such that for any $g \in Z(M)$ with $d_0(f, g) < \delta$ and any $x \in M$ there exists a unique point $y \equiv y(g, x)$ in $M$ satisfying

$$d(f^n(x), g^n(y)) < \varepsilon$$

for all $n \in \mathbb{Z}$. Let $g : M \to M$ be a homeomorphism with $d_0(f, g) < \delta$, and let $x \in M$. Then there exists a unique point $h(x)$ whose $g$-orbit $\varepsilon$ traces $\{f^n(x) : n \in \mathbb{Z}\}$. This defines an injective map $h : M \to M$ with $h \circ f = g \circ h$ and $d_0(h, 1_d) < \varepsilon$. In fact, for any $x \in M$ we have

$$d(f^n(f(x)), g^n(h(f(x)))) < \varepsilon \quad \text{and} \quad d(f^{n+1}(x), g^{n+1}(h(x))) < \varepsilon$$

for all $n \in \mathbb{Z}$. By the uniqueness, we get $hf(x) = gh(x)$.

Now we show that the map $h$ is continuous. First we claim that the map $g$ is expansive on the set $h(M)$ with an expansive constant $\frac{e}{3}$. To show this, we suppose

$$d(g^k(h(x)), g^k(h(y))) < \frac{e}{3}$$

for $x, y \in M$ and all $k \in \mathbb{Z}$. Then we have

$$\begin{aligned} d(f^k(x), f^k(y)) &\le d(f^k(x), g^k(h(x))) + d(g^k(h(x)), g^k(h(y))) \\ &\quad + d(g^k(h(y)), f^k(y)) < \varepsilon + \frac{e}{3} + \varepsilon < e \end{aligned}$$

for all $k \in \mathbb{Z}$. Since $f$ is expansive we get $x = y$, and so we have $h(x) = h(y)$.

Next we show that for any $\lambda > 0$ there exists $n \ge 1$ such that

$$d(g^k(x), g^k(y)) \le \frac{e}{3} \quad \text{for all} \quad |k| < n \quad \text{implies} \quad d(x, y) < \lambda,$$

where $x, y \in h(M)$. Note that $h(M)$ need not be compact, and hence we cannot adapt the result([7], Lemma 2) directly.

Suppose the above result does not hold. Then we can choose $\lambda > 0$ such that for each $n \ge 1$ there exists two points $x_n, y_n$ in $M$ satisfying

$$d(g^k(h(x_n)), g^k(h(y_n))) \le \frac{e}{3} \quad \text{and} \quad d(h(x_n)), h(y_n)) \ge \lambda$$

for all $|k| \le n$. Since $M$ is compact, there exist subsequences $\{x_{n_i}\}$, $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$\{x_{n_i}\} \to x \quad \text{and} \quad \{y_{n_i}\} \to y.$$

Choose a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$h(x_{n_{i_j}}) \to z \quad \text{and} \quad h(y_{n_{i_j}}) \to w, \quad \text{as} \quad j \to \infty.$$

Then for any given $k > 0$ we have

$$\begin{aligned} d(f^k(x), g^k(z)) &\le d(f^k(x), f^k(x_{n_{i_j}})) + d(f^k(x_{n_{i_j}}), g^k(h(x_{n_{i_j}}))) \\ &\quad + d(g^k(h(x_{n_{i_j}})), g^k(z)) \le \varepsilon \end{aligned}$$

for sufficiently large $j$. This means that $h(x) = z$, and so we have $h(x_{n_{i_j}}) \to h(x)$. Similarly we get $h(y_{n_{i_j}}) \to h(y)$. Consequently we obtain

$$d(g^k(h(x)), g^k(h(y))) \leq \frac{e}{3} \quad \text{and} \quad d(h(x), h(y)) \geq \lambda$$

for all $k \in \mathbb{Z}$. This contradicts the fact that $g$ in expansive on $h(M)$ with an expansive constant $\frac{e}{3}$.

To show that $h$ is continuous, we let $\lambda > 0$ be arbitrary, and let $n \geq 1$ be such that

$$d(g^k(h(x)), g^k(h(y))) \leq \frac{e}{3} \quad \text{for all} \quad |k| < n$$

implies

$$d(h(x), h(y)) < \lambda.$$

Choose $\eta > 0$ such that

$$d(x, y) < \eta \quad \text{implies} \quad d(f^k(x), f^k(y)) < \frac{e}{6}$$

for all $|k| < n$. Then we have

$$\begin{aligned} d(g^k(h(x)), g^k(h(y))) &\leq d(g^k(h(x)), f^k(x)) + d(f^k(x), f^k(y)) \\ &\quad + d(f^k(y), g^k(h(y))) < \varepsilon + \frac{e}{6} + \varepsilon < \frac{e}{3} \end{aligned}$$

for all $|k| < n$.

This means that $d(h(x), h(y)) < \lambda$. If $\varepsilon$ is sufficiently small then $d(h, 1_d) < \varepsilon$ implies that $h$ maps $M$ onto $M$. This means that $f$ is topologically stable. □

In[5], we can see that if a homeomorphism $f : M \to M$ has the SUP, then $f$ is expansive. In the process of proof of the above theorem, we showed that a homeomorphism $f : M \to M$ is strongly expansive if it has the ISUP.

COLOLLARY 10. *Any homeomorphism $f : M \to M$ is strongly expansive if it has the ISUP.*

THEOREM 11. *If $f : M \to M$ is an expansive homeomorphism with the inverse shadowing property then it is topologically stable.*

*Proof.* Let $f : M \to M$ be an expansive homeomorphism with the inverse shadowing property. Let $e > 0$ be an expansive constant of $f$ and $\varepsilon > 0$ a constant with $\varepsilon < \frac{e}{24}$. Then there exists $\delta > 0$ such that if $d_0(f, g) < \delta$ then for any $x \in M$ there is a $y \in M$ satisfying

$$d(f^k(x), g^k(y)) < \varepsilon$$

for all $k \in \mathbb{Z}$. Choose $g \in Z(M)$ with $d_0(f, g) < \delta$. Put $M/f = \{O(f, x) : x \in M\}$. Let $\alpha : M/f \to M$ be a choice function, and let $\alpha(O(f, x)) = \bar{x}$ for any $x \in M$. For each $\bar{x} \in M$, we let

$$\varepsilon(\bar{x}) = \{y \in M : d(f^k(\bar{x}), g^k(y)) < \varepsilon \ \text{ for } \ \text{all } \ k \in \mathbb{Z}\}.$$

Let $\beta : \{\varepsilon(\bar{x}) : x \in M\} \to M$ be another choice function, and let $\beta(\varepsilon(\bar{x})) = \bar{y}$ for $x \in M$. Define a map $h : M \to M$ by

$$h(f^k(\bar{x})) = g^k(\bar{y})$$

for $x \in M$ and all $k \in \mathbb{Z}$. Then $h$ is a well-defined injective map satisfying

$$d_0(h, 1_d) < \varepsilon \quad \text{and} \quad h \circ f = g \circ h.$$

In fact, for any $x_0 \in M$, we let $\alpha(O(f, x_0)) = \bar{x}_0$. Then there exists $n \in \mathbb{Z}$ with $f^n(\bar{x}_0) = x_0$. Let $\beta(\varepsilon(\bar{x}_0)) = \bar{y}_0$. Then we have

$$d(f^k(\bar{x}_0), g^k(\bar{y}_0)) < \varepsilon$$

for all $k \in \mathbb{Z}$. Hence we get

$$d(h(f^k(\bar{x}_0)), f^k(\bar{x}_0)) = d(g^k(\bar{y}_0), f^k(\bar{x}_0)) < \varepsilon$$

for all $k \in \mathbb{Z}$. This means that $d(h, 1_d) < \varepsilon$. Moreover we obtain

$$\begin{aligned} hf(x_0) &= hf(f^n(\bar{x}_0)) = h(f^{n+1}(\bar{x}_0)) \\ &= g^{n+1}(\bar{y}_0) = g(g^n(\bar{y}_0)) \\ &= g(hf^n(\bar{x}_0)) = gh(x_0). \end{aligned}$$

.

To show that $h$ is injective, we suppose that $h(x) = h(x_0)$ for $x, x_0 \in M$. Since $h \circ f = g \circ h$, we have

$$\begin{aligned} d(f^k(x), f^k(x_0)) &\le d(f^k(x), hf^k(x)) + d(hf^k(x_0), f^k(x_0)) \\ &< \varepsilon + \varepsilon < e, \end{aligned}$$

for all $k \in \mathbb{Z}$. This means that $x = x_0$.

Now we show that the map $h$ is continuous. First we can easily check that $g$ is expansive on $h(M)$ with an expansive constant $\frac{e}{3}$. Suppose that we can choose $\lambda > 0$ such that for any $n \ge 1$ there exist two points $x_n, y_n \in M$ satisfying

$$d(g^k(h(x_n)), g^k(h(x_n))) \le \frac{e}{6} \quad \text{and} \quad d(h(x_n), h(y_n)) \ge \lambda$$

for all $|k| < n$. Since $M$ is compact, there exist subsequences $\{x_{n_i}\}$, $\{y_{n_i}\}$ of $\{x_n\}$ and $\{y_n\}$, respectively, such that

$$\{x_{n_i}\} \to x \quad \text{and} \quad \{y_{n_i}\} \to y.$$

Choose a subsequence $\{n_{i_j}\}$ of $\{n_i\}$ such that

$$h(x_{n_{i_j}}) \to z \quad \text{and} \quad h(y_{n_{i_j}}) \to w, \quad \text{as} \quad j \to \infty.$$

Then we have

$$d(g^k(z), g^k(w)) \leq \frac{e}{6} \quad \text{for all} \quad k \in \mathbb{Z}, \text{ and } \ d(z, w) \geq \lambda.$$

And for any given $k > 0$ we have

$$\begin{aligned} d(f^k(x), g^k(z)) \leq d(f^k(x), f^k(x_{n_{i_j}})) + d(f^k(x_{n_{i_j}}), g^k(h(x_{n_{i_j}}))) \\ + d(g^k(h(x_{n_{i_j}})), g^k(z)) \leq \varepsilon, \end{aligned}$$

for sufficiently large $j$. Similarly we get

$$\begin{aligned} d(f^k(y), g^k(w)) \leq d(f^k(y), f^k(y_{n_{i_j}})) + d(f^k(y_{n_{i_j}}), g^k(h(y_{n_{i_j}}))) \\ + d(g^k(h(y_{n_{i_j}})), g^k(w)) \leq \varepsilon. \end{aligned}$$

Therefore we have

$$\begin{aligned} d(g^k(h(x)), & g^k(h(y))) \\ & \leq d(g^k(h(x)), f^k(x)), + d(f^k(x), g^k(z)), \\ & + d(g^k(z), g^k(w)) + d(g^k(w), f^k(y)), + d(f^k(y), g^k(h(y))) \\ & \leq \varepsilon + \varepsilon + \frac{e}{6} + \varepsilon + \varepsilon \leq \frac{e}{3}, \end{aligned}$$

for all $k \in \mathbb{Z}$. Consequently we obtain

$$d(g^k(h(x)), g^k(h(y))) \leq \frac{e}{3} \quad \text{and} \quad d(h(x), h(y)) > 0$$

for all $k \in \mathbb{Z}$. This contradicts the fact that $g$ is expansive on $h(M)$ with an expansive constant $\frac{e}{3}$.

The fact that the map $h$ is continuous can be proved by the same techniques as in the proof of theorem 9, and so $f$ is topologically stable. □

COROLLARY 12. *If $f : M \to M$ is an expansive homeomorphism with the inverse shadowing property then it has the shadowing property.*

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