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STABILITY FOR INTEGRO–DELAY–DIFFERENTIAL EQUATIONS

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ABSTRACT. We will investigate some properties of integro-delay- differential equations,

$$\begin{aligned} x'(t) =& A(t)x(t-g_1(t,x_t)) + \int_{t_0}^{t} B(t,s)x(s-g_2(s,x_s))ds, \quad t_0 \geq 0, \\ x(t_0) =& \phi, \end{aligned}$$

1. Introduction

In this paper, we concentrate on a system of integro-delay-differential equations,

$$\begin{aligned} x'(t) =& A(t)x(t-g_1(t,x_t)) \\ & + \int_{t_0}^{t} B(t,s)x(s-g_2(s,x_s))ds, \qquad t_0 \geq 0, \\ x(t_0) =& \phi. \end{aligned} \tag{1}$$

where $A(t)$ and $B(t,s)$ are continuous $n \times n$ matrices on $\mathbb{R}^+$ and $\mathbb{R}^+ \times \mathbb{R}^+$,$\mathbb{R}^+ = [0,\infty)$ respectively, and $g_i : \mathbb{R}^+ \times C_n^q \to [0,q], i = 1,2,$ is continuous and $C_n^q = C([-q,0],\mathbb{R}^n), q > 0.$

For a delay differential equations

$$x'(t) = A(t)x(t - r(t,x_t),) \tag{2}$$

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Yoneyama [6] has been studied the stability property for (2) by means of the variation of constant formula. It was given by

$$x(t) = \Phi(t,t_0)x(t_0) + \int_{t_0}^{t} \Phi(t,s)A(s)(x(s - r(s,x_s) - x(s))ds,$$

where $\Phi(t,t_0) = \Phi(t)\Phi(t_0)^{-1}$ is the fundamental matrix solution of $x'(t) = A(t)x(t)$. Moreover, he obtained some examples[6].

Here we will investigate the exponential asymptotic stability and h-stability for (1) by means of resolvent matrix solutions.

2. Preliminaries

We consider a system of functional differential equations

$$x'(t) = f(t, x_t) \tag{3}$$

where $f : \mathbb{R}^+ \times C_n^q \to \mathbb{R}^n$ is continuous, $f(t,0) = 0$. We denote $x_t \in \mathcal{C}_n^q$, the function defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in [-q,0]$. Set $||\phi|| = \sup_{\theta \in [-q,0]} |\phi(\theta)|$ for $\phi \in C_n^q$, where $|\cdot|$ is denotes arbitrary vector norm in $\mathbb{R}^n$. Let $x = x(t_0,\phi,f)$ be the unique solution of (3) with initial function ϕ such that $x_{t_0} = \phi$.

The value of $x(t_0,\phi,f)$ at t is denoted by $x(t) = x(t_0,\phi,f)(t)$ and $x = 0$ is called a zero solution.

Definition 2.1. The zero solution $x = 0$ of (3) is said to be *exponentially asymptotically stable*{EAS} if for any $t \geq t_0 \geq 0$, there exist positive constant c and M such that

$$|x(t)| \leq M||\phi||e^{-c(t-t_0)}$$

provided $||\phi|| < \delta$ for some $\delta > 0$.

Definition 2.2. The zero solution $x = 0$ of (3) is said to be h-stable{hS} if there exist $M \geq 0$, $\delta > 0$ and a positive bounded continuous function h on $\mathbb{R}^+$ such that

$$|x(t)| \leq M||\phi||h(t)h(t_0)^{-1}$$

provided $||\phi|| < \delta$ for $t \geq t_0 \geq 0$,

Consider the linear integro-differential system

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)ds. \tag{4}$$

The resolvent matrix solution of (5), $R(t,s)$, is given by

$$\frac{\partial}{\partial s}R(t,s) + R(t,s)A(s) + \int_{s}^{t} R(t,u)B(u,s)ds = 0, \tag{5}$$

with $R(t,t) = I$, the identity matrix for $0 \leq s \leq t < \infty$ [4].

We can rewrite (1) in the form

$$x'(t) = A(t)x(t) + \int_{t_0}^{t} B(t,s)x(s)ds + G(t,x_t) \tag{6}$$

where

$$\begin{aligned} G(t,x_t) =& A(t)(x(t-g_1(t,x_t)) - x(t)) \\ &+ \int_{t_0}^{t} B(t,s)(x(s-g_2(s,x_s)) - x(s))ds. \end{aligned} \tag{7}$$

Then the unique solution of (1) is given by

$$x(t) = R(t,t_0)x(t_0) + \int_{t_0}^{t} R(t,s)G(s,x_s)ds, \quad t \geq t_0 \geq 0. \tag{8}$$

where $R(t,s)$ satisfies (5) [4].

We suppose that the following condition:

(H_1) there exists a constant $\varepsilon > 0$ such that $g_i(t,\phi) \leq \varepsilon||\phi||, i = 1,2$.

(H_2) there exists a constant K such that

$$\sup_{t_0 \leq s \leq t}(|A(s)| + e^{-cs}\int_{s}^{t} e^{c\tau}|B(\tau,s)|d\tau) \leq K.$$

(H_3) there exists positive constant L such that

$$\sup_{t-g_i(t,x_t) \leq \xi_i \leq t} |x'(\xi_i)| \leq L, i = 1,2.$$

3. Main Results

LEMMA 3.1. *For any $t_0 \geq 0$ and $\phi \in C_n^q$, the solution x of (1) is defined for all $t \geq t_0$ and*

$$|x(t)| \leq ||\phi|| e^{\int_{t_0}^t (|A(s)| + \int_s^t |B(\tau,s)| d\tau)} ds. \tag{9}$$

Proof. By using Fubini's theorem, we obtain

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^t A(s) x(s - g_1(s, x_s)) ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s B(s,\tau) x(\tau - g_2(\tau, x_\tau)) d\tau ds \\ &= \phi + \int_{t_0}^t A(s) x(s - g_1(s, x_s)) ds \\ &\quad + \int_{t_0}^t \int_s^t B(\tau, s) d\tau x(s - g_2(s - x_s)) ds. \end{aligned}$$

Since $|x(t - g_i(t, x_t))| \leq ||x_t||$,

$$|x(t)| \leq ||\phi|| + \int_{t_0}^t (|A(s)| + \int_s^t |B(\tau, s)| d\tau) ||x_s|| ds.$$

Then Gronwall's inequality implies inequality (9). □

LEMMA 3.2. [2] *The zero solution of (4) EAS if and only if there exist constant $c > 0$ and $M > 0$ such that $|R(t, t_0)| \leq M e^{-c(t - t_0)}$.*

THEOREM 3.3. *Under the hypotheses H_1, H_2, H_3, if the zero solution of (4) is EAS, then the zero solution of (1) EAS whenever $c > MLK\varepsilon$.*

Proof. From (8) we have

$$\begin{aligned} |x(t)| \leq & |R(t, t_0)| ||\phi|| + \int_{t_0}^t |R(t, s)| |F(s, x_s)| ds \\ \leq & M ||\phi|| e^{-c(t - t_0)} + M \int_{t_0}^t e^{-c(t-s)} |A(s)| |x(s - g_1(s, x_s)) - x(s)| ds \\ & + M \int_{t_0}^t e^{-c(t-s)} \int_{t_0}^s |B(s, \tau)| |x(\tau - g_2(\tau, x_\tau)) - x(\tau)| d\tau ds. \end{aligned}$$

Since $x(t-g_i(t,x_t))-x(t)=\int_t^{t-g_i(t,x_t)}x'(s)ds$,

$$\begin{aligned}|x(t)|\leq & M||\phi||e^{-c(t-t_0)}+LM\int_{t_0}^{t}e^{-c(t-s)}|A(s)|g_1(x,x_s)ds\\ &+LM\int_{t_0}^{t}e^{-c(t-s)}\int_{t_0}^{s}|B(s,\tau)|g_2(\tau,x_\tau)d\tau ds.\end{aligned}$$

By changing the order of integration,

$$\begin{aligned}|x(t)|\leq & M||\phi||e^{-c(t-t_0)}+LM\int_{t_0}^{t}e^{-c(t-s)}|A(s)|\varepsilon||x_s||ds\\ &+LM\int_{t_0}^{t}e^{-ct}\int_{s}^{t}e^{c\tau}|B(\tau,s)|d\tau\varepsilon||x_s||ds.\end{aligned}$$

Then

$$\begin{aligned}e^{ct}||x_t||\leq & M||\phi||e^{ct_0}\\ &+ML\varepsilon\int_{t_0}^{t}(|A(s)|+e^{-cs}\int_{s}^{t}e^{c\tau}|B(\tau,s)|d\tau)||x_s||e^{cs}ds.\end{aligned}$$

Therefore, from Gronwall's inequality, we obtain

$$||x_t||\leq M||\phi||e^{-(c-MLK\varepsilon)(t-t_0)}$$

The proof of the theorem is completed. □

Corollary 3.4. *Under the hypotheses in Theorem 3.3, if $c>\varepsilon MLK$ then the zero solution of (1)tends to zero exponentially as $t\to\infty$. Also, if $c=\varepsilon MLK$, then the zero solution of (1) is uniformly stable.*

Denote $\tilde{h}(t)=h(t)$ for $t\geq 0$ and $\tilde{h}(t)=h(0)$ for $t<0$. We impose another hypothesis:

$$\begin{aligned}&\lambda(t,s)=\sup_{t_0\leq s\leq t}||\tilde{h}_s||\{|A(s)|h(s)^{-1}\\ (H_4)\qquad &\qquad+\int_{s}^{t}h(\tau)^{-1}|B(\tau,s)|d\tau\},\int_{t_0}^{\infty}\lambda(t,s)ds<\infty.\end{aligned}$$

LEMMA 3.5. [1] *The zero solution of (4) h-stable if and only if there exist constant $M > 0, \delta > 0$ and a positive bounded continuous function h defined on $\mathbb{R}^+$ such that*

$$|R(t,t_0)| \leq Mh(t)h(t_0)^{-1}$$

for $t \geq t_0 \geq 0$ and $||\phi|| \leq \delta$.

THEOREM 3.6. *Under the hypotheses H_1, H_3, H_4, if the zero solution of (4) is h-stable, then the zero solution of (1) h-stable*

Proof. By the same method as Theorem 3.3, we have

$$\begin{aligned}|x(t)| \leq & M|x(t_0)|h(t)h(t_0)^{-1} \\ & + ML\varepsilon h(t)\{\int_{t_0}^t h(s)^{-1}|A(s)|||x_s||ds \\ & + \int_{t_0}^t \int_s^t h(\tau)^{-1}|B(\tau,s)|d\tau||x_s||ds\}.\end{aligned}$$

Set $|z(t)| = h(t)^{-1}|x(t)|$. Then

$$\begin{aligned}|z(t)| \leq & M|z(t_0)| + ML\varepsilon\{\int_{t_0}^t |A(s)|h(s)^{-1}||\tilde{h}_s||||z_s||ds \\ & + \int_{t_0}^t \int_s^t h(\tau)^{-1}|B(\tau,s)|d\tau||\tilde{h}_s||||z_s||\}ds \\ \leq & M|z(t_0)| + ML\varepsilon \int_{t_0}^t ||\tilde{h}_s||\{|A(s)|h(s)^{-1} \\ & \cdot \int_s^t h(\tau)^{-1}|B(\tau,s)|d\tau\}||z_s||ds.\end{aligned}$$

Theofore

$$||z_t|| \leq M||z_{t_0}|| + ML\varepsilon \int_{t_0}^t \lambda(t,s)||z_s||ds.$$

By Gronwall's inequality we obtain

$$||z_t|| \leq ||z_{t_0}||M_1,$$

where $M_1 = Me^{MLε \int_{t_0}^{\infty} λ(t,s)ds}$ for all $t \geq t_0$. Therefore

$$\begin{aligned} |x(t)| =& h(t)|z(t)| \\ \leq& h(t)||z_t|| \\ \leq& M_1 h(t)||z_{t_0}|| \\ \leq& M_1 h(t)||x_{t_0}||||\tilde{h}_{t_0}|| \\ =& M_2 M_1 ||\phi|| h(t) h(t_0)^{-1}, \end{aligned}$$

where $M_2 = h(t_0)||\tilde{h}_{t_0}||$ for $t \geq t_0 \geq 0$.The proof is complete. □

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